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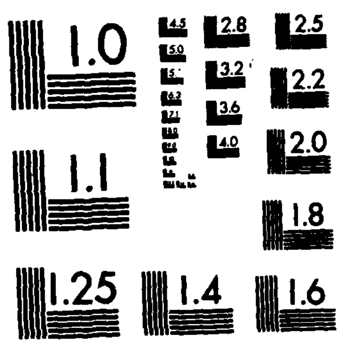

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ABSTRACT

This report investigates the idea of utilizing Luenberger's minimal-order observer as an alternate to the Kalman filter for obtaining state estimates in linear discrete-time stochastic systems. More specifically, this dissertation presents a solution to the problem of constructing an optimal minimal-order observer for linear discrete-time stochastic systems where optimality is in the mean-square sense. The approach taken in this dissertation leads to a completely unified theory for the design of optimal minimal-order observers and is applicable to both time-varying and time-invariant linear discrete systems. The basic solution to the problem is first obtained for that class of systems having Gaussian white noise disturbances. The solution is based on a special linear transformation which transforms the given time-varying discrete-time state equations into an equivalent state space which is extremely convenient from the standpoint of observer design. Design of the observer is based on a special observer configuration containing a free gain matrix, K_i, which is chosen to minimize the mean-square estimation error at time "i." The solution obtained is optimal at each instant "i" and therefore is optimal both during the transient period and in the steady-state. Computation of this optimal gain matrix is recursive as in the Kalman filter algorithms, however, computationally the solution is much simpler than for the Kalman filter. In the special case of no measurement noise, the observer estimation errors are identical with those of the corresponding Kalman filter. When measurement noise is not excessive, estimation errors comparable with a Kalman filter are obtained.

Next, the basic observer solution is extended to the class of systems for which the noise disturbances are time-wise correlated processes of the Gauss-Markov type. In considering the correlated noise inputs, the basic observer structure is used directly, i.e., it is not necessary to augment the plant state equations as is done in the usual Kalman filtering theory. The observer free gain matrix, K_i, is modified appropriately to account for the time-wise correlation of the noise inputs and is chosen again to yield minimum mean-square error estimates of the state vector.

To illustrate the theory and application of the observer designs developed in the dissertation the problem of designing a radar tracking system is considered. Examples are included which illustrate clearly the practicality and usefulness of the proposed optimal observer design technique.

Finally, a host of topics for future research is presented in the hope of stimulating further research in the domain of observer theory.

ADVANCES IN OBSERVER TECHNIQUES FOR BALLISTIC MISSILE DEFENSE FILTERING ALGORITHMS

by

Leslie M. Novak

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1. INTRODUCTION AND OUTLINE OF RESEARCH

1.1 INTRODUCTION AND PROBLEM STATEMENT

The general state estimation problem to be considered in this dissertation is described simply as follows. Given the linear stochastic discrete-time dynamical system characterized by the equations

$$\underline{x}_{i+1} = A_i \underline{x}_i + \underline{w}_i \tag{1.1}$$

$$\underline{y}_i = H_i \underline{x}_i + \underline{v}_i \tag{1.2}$$

where

$\underline{x}_i$ is the n-dimensional state vector

$\underline{y}_i$ is the m-dimensional measurement vector

$\underline{w}_i$ and $\underline{v}_i$ are, respectively, n-dimensional and m-dimensional independent Gaussian white noise sequences having zero means and covariances Q_i and R_i

and $\underline{x}_o$, the initial state, is an independent Gaussian vector with mean $\overline{\underline{x}}_o$ and covariance M_o. It is desired to find an estimate of the state vector $\underline{x}_i$ at time "i" along with its associated error covariance $P_{i/i}$. The notation $\hat{\underline{x}}_{i/i}$ implies the estimate is to be based on all the measurements obtained up to and including $\underline{y}_i$ obtained at time "i." It is, of course, also desired that this estimate $\hat{\underline{x}}_{i/i}$ be optimal in some sense, i.e., with respect to some given performance criterion. There are many performance criteria which have been presented in the literature pertaining to estimation theory, however, from the standpoint of mathematical tractability the quadratic

performance criterion is most appealing and it was this performance criterion which was used quite successfully by Kalman [14]. If it is desired that the estimate $\hat{\underline{x}}_{i/i}$ be optimal in the mean-square sense, which implies that the estimate $\hat{\underline{x}}_{i/i}$ minimizes the quantity $E\{\|\hat{\underline{x}}_{i/i}-\underline{x}_i\|^2\}$ then the solution to the estimation problem is the well-known Kalman filter and the defining equations for the optimal Kalman estimator are

$$\hat{\underline{x}}_{i+1/i+1} = \hat{\underline{x}}_{i+1/i} + K_{i+1}(\underline{y}_{i+1} - H_{i+1}\hat{\underline{x}}_{i+1/i}) \tag{1.3}$$

$$K_{i+1} = P_{i+1/i}H_{i+1}'(H_{i+1}P_{i+1/i}H_{i+1}' + R_{i+1})^{-1} \tag{1.4}$$

$$P_{i+1/i} = A_i P_{i/i} A_i' + Q_i \tag{1.5}$$

$$P_{i+1/i+1} = (I_n - K_{i+1}H_{i+1})P_{i+1/i} \tag{1.6}$$

where $\hat{\underline{x}}_{i+1/i} = A_i\hat{\underline{x}}_{i/i}$. To initialize the Kalman filter at time "i=0" we take $\hat{\underline{x}}_{o/o} = \overline{\underline{x}}_o$ and $P_{o/o} = M_o$.

Although in theory the Kalman filter completely solves the problem of state estimation in the mean-square sense for linear systems with Gaussian statistics, its inherent complexity and implementation have discouraged widespread application. Building the Kalman filter essentially requires the simulation of the entire n-dimensional system being observed. Equally important, the Ricatti equations (1.5) and (1.6) which must be solved at each time instant "i" to obtain the optimal Kalman gain matrix, K_i, have been the source of much trouble in the real-time mechanization of Kalman filters, especially in the case of large dimensional systems. These numerical and computational problems associated with the real-time implementation of Kalman filters have led many researchers to seek out

simpler, less optimal solutions to the minimum mean-square state estimation problem.

Early work in this area was done by Luenberger [20-22] who showed that when the system (1.1) and (1.2) is time-invariant and no noise disturbances are present, the state vector $\underline{x}_i$ may be reconstructed exactly with a stable linear system of order "n-m" which he called a minimal-order observer. Luenberger's basic idea in the development of his minimal-order observer is the notion that since there are "m" independent measurements already available it should be possible to reconstruct the entire n-dimensional state vector of the system by generating only "n-m" additional quantities and combining them appropriately with the "m" already existing outputs. Of course, Luenberger's basic assumption that the system inputs are free of noise is not always satisfied in practice and this comprises a fundamental limitation to his original work.

Next, Aoki and Huddle [3] extended Luenberger's work to include the effects of noise disturbances $\underline{w}_i$ and $\underline{v}_i$. However, their work was restricted to time-invariant systems and as a result their technique is not directly applicable to the more general time-varying system modeled in (1.1) and (1.2). The technique presented in Aoki and Huddle [3] was essentially to construct a minimal-order observer which minimized the steady-state mean-square estimation error. However, their optimization technique is computationally formidable, even for the simplest of systems, and as a result does not appear to have been used to any large degree in the design of minimal-order observers for practical engineering systems.

Attempts to construct optimal observer designs based on a purely deterministic point of view also appear to have been fruitless. Newmann [23]

has investigated the standard optimal control problem with a quadratic cost function for the case of linear time-invariant systems using an observer in the feedback path when some of the state variables are not measurable. By counterexample, he clearly demonstrated that if nothing is known about the initial conditions of the state vector then there is no way of designing the observer so that the cost of control will be minimized. In fact, if nothing is known about the initial conditions then high cost may result from the use of an observer in the feedback path.

Dellon [10] also studied the deterministic feedback optimal control problem with the standard quadratic cost function from the standpoint of using a minimal-order observer in the feedback to reconstruct the state vector $\underline{x}_i$. Dellon considered the more general time-varying discrete system in the absence of noise disturbances and has indicated similar findings. Restricting his observer design to that class of observers having constant and equal eigenvalues he concluded that the relative degradation in cost from the optimal (i.e., when all the states are available for feedback) cannot be made arbitrarily small by proper choice of observer eigenvalues but the relative degradation depends upon the original optimization problem.

More recently, Ash [4, 5] developed a sub-optimal minimal-order observer estimator design applicable to both discrete and continuous time-varying stochastic systems. His main goal was to develop a stable minimal-order observer which provided "acceptable" mean-square estimation errors. Ash himself stated that his work comprises an engineering solution rather than a mathematical solution to the problem. The design procedure of Ash is a "trial and error" technique which, if judiciously applied, may result in a relatively good sub-optimal estimator in comparison to the corresponding

optimal Kalman filter. However, in the utilization of Ash's "trial and error" technique it is not at all clear how to achieve acceptable performance without trying many designs and selecting the best design out of those which were tried out.

To review the preceeding paragraphs, we have introduced the fundamental problem of minimum mean-square estimation for linear discrete stochastic systems and have indicated Kalman's optimal solution under the assumption of Gaussian noise processes. After describing Kalman's filter and its inherent problems of computation and implementation in real-time systems, we next considered the idea of using Luenberger's minimal-order observer as an alternate to the Kalman filter. The evolution of Luenberger's basic observer theory is then presented through a discussion of the attempts of various researchers to design observers which are optimal in some sense, both from a deterministic control theory point of view as well as from a more general stochastic estimation theory point of view. Through this evolutionary discussion we have attempted to provide the reader with a smooth transition from Luenberger's original concept of a minimal-order observer to the ultimate topic of this dissertation. It should be clear from the historical evolution that the solution for an optimal minimal-order observer has importance not only from a theoretical standpoint but also from the standpoint of designing optimal and suboptimal engineering systems. For these reasons, we have considered, in this dissertation, the problem of constructing an optimal minimal-order observer for discrete-time stochastic systems and, in the spirit of Kalman, have chosen the mean-square estimation error as our performance criterion.

1.2 OUTLINE OF THE DISSERTATION

Chapter 2 is a presentation of some of the more important basic results of observer theory as related to deterministic discrete time-varying systems. Chapter 2 has been included mainly for completeness and is intended to introduce the reader to the basics of observer theory. Those familiar with the material may skip Chapter 2 without loss of continuity.

New theoretical results are given in Chapter 3, in which is presented the fundamental solution for the optimal minimal-order observer in the case where the noises $\underline{w}_i$ and $\underline{v}_i$ are Gaussian white noise sequences. Also, in Chapter 3 the complete generality of the optimal minimal-order observer design is discussed and the equivalence of this observer and the Kalman filter is demonstrated for the special case in which the measurement noise, $\underline{v}_i$, is identically zero. Chapter 4 treats important new extensions of the basic minimal-order observer design to the class of systems in which the noise disturbances $\underline{w}_i$, $\underline{v}_i$ are time-wise correlated processes of the Gauss-Markov type.

A comprehensive and comparative study of several observer designs, including the Kalman filter, the optimal minimal-order observer, and several equal eigenvalue observer designs, is presented in the examples of Chapter 5. The computer simulations of Chapter 5 treat the practical problem of designing a radar tracking system of reduced complexity based on the optimal minimal-order observer solutions developed in the previous chapters 3 and 4 of the dissertation.

The final conclusions and recommendations for further research are presented in Chapter 6.

2. SOME FUNDAMENTAL RESULTS OF DETERMINISTIC OBSERVER THEORY

2.1 MINIMAL-ORDER OBSERVERS FOR DETERMINISTIC SYSTEMS

The purpose of this chapter is to review some of the more important fundamental results of deterministic observer theory which have been obtained by various researchers to date. We begin by defining the concept of a minimal-order observer for linear discrete-time dynamical systems. Huddle [13] has shown that a completely observable n-dimensional system

$$\underline{x}_{i+1} = A_i\underline{x}_i + B_i\underline{u}_i \tag{2.1}$$

with m independent outputs

$$\underline{y}_i = H_i\underline{x}_i \tag{2.2}$$

can be "observed" with an (n-m)-dimensional system

$$\underline{z}_{i+1} = F_i\underline{z}_i + G_i\underline{u}_i + D_i\underline{y}_i \tag{2.3}$$

such that the output of the observer is of the form

$$\underline{z}_i = T_i\underline{x}_i + \underline{\varepsilon}_i \tag{2.4}$$

where

$$\underline{\varepsilon}_i \triangleq \left(\prod_{j=o}^{i-1} F_j\right)(\underline{z}_o - T_o\underline{x}_o) \tag{2.5}$$

If the observer initial condition is chosen such that $\underline{z}_o = T_o\underline{x}_o$ then from (2.4), (2.5) it is seen that $\underline{z}_i = T_i\underline{x}_i$ for all "i" ≥ 0 and in this case it is possible to reconstruct $\underline{x}_i$ exactly from $\underline{y}_i$ and $\underline{z}_i$. The observer is chosen so that the rows of $\left[\frac{T_i}{H_i}\right]$ are linearly independent and the estimate of $\underline{x}_i$ is taken as

$$\hat{\underline{x}}_i = \left[\frac{T_i}{H_i}\right]^{-1} \begin{bmatrix} \underline{z}_i \\ \underline{y}_i \end{bmatrix} \tag{2.6}$$

If $\underline{z}_o = T_o\underline{x}_o$ then (2.6) will give the true value of the state $\underline{x}_i$.

Huddle also proved that for the system (2.3) to be an observer of the state $\underline{x}_i$ in (2.1) it is both necessary and sufficient that the following matrix relations be satisfied

$$T_{i+1}A_i = F_iT_i + D_iH_i \tag{2.7}$$

$$G_i = T_{i+1}B_i \tag{2.8}$$

Further, since it is necessary that the matrix inverse $\left[\frac{T_i}{H_i}\right]^{-1}$ exist, Huddle postulated the inverse to be partitioned in the form $[P_i | V_i]$ and obtained the solution of (2.7) to be

$$F_i = T_{i+1}A_iP_i \tag{2.9}$$

$$D_i = T_{i+1}A_iV_i \tag{2.10}$$

where $P_iT_i + V_iH_i = I$.

By using a clever coordinate transformation Dellon [10] next extended the work of Huddle by proving that the eigenvalues of the observer matrix F_i are completely arbitrary provided the system (2.1) is completely uniformly observable. To do this Dellon assumed the measurement matrix

to be of the form

$$H_i = \left[H_i^{(1)} \middle| H_i^{(2)} \right] \tag{2.11}$$

where $H_i^{(1)}$ is an m x m full rank matrix at each "i". Then using the linear transformation

$$\underline{q}_i = \left[\begin{array}{c|c} I_m & -H_i^{(1)^{-1}} H_i^{(2)} \\ \hline 0 & I_{n-m} \end{array} \right] \underline{x}_i$$

Dellon obtained an equivalent state space where the measurement matrix was in the form

$$H_i = \left[H_i^{(1)} \middle| \ 0 \right] \tag{2.13}$$

Without loss of generality the system (2.1), (2.2) was assumed to be already in this desired form and the observer matrix T_i was taken to be

$$T_i = [K_i | I] \tag{2.14}$$

where K_i is a free (n-m) x m gain matrix. From (2.9) it is shown that the observer matrix F_i is of the form

$$F_i = A_{22}^{\ i} + K_{i+1} A_{12}^{\ i} \tag{2.15}$$

where $A_{22}^{\ i}$ and $A_{12}^{\ i}$ are respectively (n-m) x (n-m) and m x (n-m) partitions of the matrix A_i in (2.1). Invoking the dual of Wonham's result for controllability, [33] Dellon argued that if $(A_{22}^{\ i}, A_{12}^{\ i})$ is an observable pair then there exists a matrix K_{i+1} such that the eigenvalues of

$A_{22}^{\ i} + K_{i+1}A_{12}^{\ i}$ may be arbitrarily assigned. But $(A_{22}^{\ i}, A_{12}^{\ i})$ is an observable pair at every "i" provided the system is completely uniformly observable. Thus, the eigenvalues of F_i are completely arbitrary at each instant "i".

Returning to the idea of state reconstruction, we note that since the entire state $\underline{x}_o$ is not directly accessible it is unlikely that the condition $\underline{z}_o = T_o\underline{x}_o$ can be achieved. This implies that the observer error (2.5) will in general be non zero and the estimate $\hat{\underline{x}}_i$ in (2.6) will be in error. However, since the observer eigenvalues were shown to be completely arbitrary, it is therefore possible to reduce the observer error to zero as rapidly as desired. Thus, we have forced the estimate $\hat{\underline{x}}_i$ to approach the true state $\underline{x}_i$ as rapidly as desired.

2.2 OBSERVERS OF ORDER "n"

Williams [31] has considered non-minimal order observers and has approached the observer design problem with the idea of achieving suboptimal Kalman filtering. Consider the n-dimensional observer given as

$$\underline{z}_{i+1} = F_i\underline{z}_i + G_i\underline{u}_i + D_{i+1}\underline{y}_{i+1} \tag{2.16}$$

Here the observer output is defined by the relation

$$\underline{z}_i = T_i\underline{x}_i + \underline{\epsilon}_i \tag{2.17}$$

where T_i is an n x n nonsingular matrix. In this case the state estimate $\hat{\underline{x}}_i$ is taken to be

$$\hat{\underline{x}}_i = T_i^{-1}\underline{z}_i \tag{2.18}$$

Williams has shown that the system (2.16) is an observer of the state $\underline{x}_i$ in (2.1) if and only if the following matrix relations are satisfied

$$T_{i+1}A_i = F_iT_i + D_{i+1}H_{i+1}A_i \tag{2.19}$$

$$T_{i+1}B_i = G_i + D_{i+1}H_{i+1}B_i \tag{2.20}$$

The corresponding estimation error is given by the expression

$$\underline{e}_i = T_i^{-1}\left(\prod_{j=o}^{i-1} F_j\right)(\underline{z}_o - T_o\underline{x}_o) \tag{2.21}$$

One obtains an interesting solution to (2.19) by taking

$$D_{i+1} = T_{i+1}K_{i+1} \tag{2.22}$$

where K_{i+1} is an arbitrary n x m gain matrix. With this choice for D_{i+1} the observer equations become

$$F_iT_i = T_{i+1}(I - K_{i+1}H_{i+1})\, A_i \tag{2.23}$$

$$G_i = T_{i+1}(I - K_{i+1}H_{i+1})\, B_i \tag{2.24}$$

An interesting observation concerning (2.23), (2.24) is that the special case where $T_i = I$ and K_i is taken to be the Kalman filter gain matrix, the observer obtained is identical to the Kalman filter. That is, the observer equations become

$$F_i = (I - K_{i+1}H_{i+1})\, A_i \tag{2.25}$$

$$G_i = (I - K_{i+1}H_{i+1})\, B_i \tag{2.26}$$

$$D_{i+1} = K_{i+1} \tag{2.27}$$

Substituting (2.25), (2.26) and (2.27) into the observer system equation (2.16) gives the result

$$\underline{z}_{i+1} = A_i\underline{z}_i + B_i\underline{u}_i + K_{i+1}\left(\underline{y}_{i+1} - H_{i+1}(A_i\underline{z}_i + B_i\underline{u}_i)\right) \tag{2.28}$$

which clearly shows the observer to be identical to the Kalman filter. If the designer picks the gain matrix K_{i+1} according to some other criterion, the observer then may be viewed as a suboptimal Kalman filter. (For example, the gain matrix might be chosen to give some arbitrary set of eigenvalues.) Therefore, a Kalman filter is an n-dimensional observer for which the weighting matrix D_{i+1} has been chosen to minimize the mean square estimation error. It is also interesting to note that in the more general case where the transformation T_i is a k x n rectangular matrix ($k < n$), the solution of the fundamental observer equation (2.23) is an aggregation in the sense of Aoki.[2] We shall not pursue this idea any further since our interest in this observer formulation will be primarily the design of n-dimensional observers based on the selection of eigenvalues.

By a judicious choice for the observer transformation T_i, Williams has shown that it is possible to obtain completely arbitrary eigenvalues at each instant "i" for an observer of the form (2.16). He considered a completely uniformly observable pair $(A_i, \underline{h}_i)$ and took as the transformation T_i the following matrix product

$$
T_i \triangleq \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\lambda_n^{i-1} & 1 & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -\lambda_3^{i-1} & & & 1 & 0 \\ -\lambda_2^{i-1} & -\lambda_3^{i-1} & \cdots & -\lambda_n^{i-1} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \underline{h}_{i+1} \prod_{j=0}^{0} A_{i+j} \\ \underline{h}_{i+1} \prod_{j=0}^{1} A_{i+j} \\ \vdots \\ \underline{h}_{i+1} \prod_{j=0}^{n-1} A_{i+j} \end{bmatrix}}_{\text{Observability Matrix}} \tag{2.29}
$$

where for the purpose of simplicity we have considered a single output system. The results are easily extended to the multiple output case. For the particular transformation T_i chosen the observer system matrix F_i is in column companion form and has arbitrary eigenvalues.

$$
F_i = \underbrace{\begin{bmatrix} \lambda_n^i & 1 & 0 & \cdots & 0 \\ \lambda_{n-1}^i & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ \lambda_2^i & & & & 1 \\ \lambda_1^i & 0 & 0 & \cdots & 0 \end{bmatrix}}_{T_{i+1}A_iT_i^{-1}} - \underbrace{\begin{bmatrix} x \\ x \\ \vdots \\ x \end{bmatrix}}_{T_{i+1}K_{i+1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\underline{h}_{i+1}A_iT_i^{-1}} \tag{2.30}
$$

Since the gain matrix K_{i+1} is completely arbitrary and the matrix T_{i+1} is nonsingular, from (2.30) it is apparent that any desired set of observer eigenvalues may be obtained.

2.3 APPLICATION TO OPTIMAL CONTROL

One of the fundamental applications of observer theory is in the design of feedback controllers for the linear regulator problem where some

of the states are inaccessible and must therefore be estimated using an observer. For example, assume it is required to obtain the control $\underline{u}_i$ in (2.1) which minimizes the cost function

$$J = \sum_{i=o}^{N} \underline{x}_i{}'Q_i\underline{x}_i + \underline{u}_i{}'R_i\underline{u}_i \tag{2.31}$$

where Q_i and R_i are respectively n x n and p x p symmetric positive definite matrices for all "i" in the interval $[0,N]$. The feedback law which minimizes J is known to be a linear state feedback of the form [30]

$$\underline{u}_i = \Lambda_i\underline{x}_i \tag{2.32}$$

where

$$\Lambda_i \stackrel{\Delta}{=} -(R_i + B_i{}'\Gamma_{i+1}B_i)^{-1} B_i{}'\Gamma_{i+1}A_i \tag{2.33}$$

and Γ_i is the n x n symmetric positive definite solution to the discrete Ricatti equation

$$\Gamma_i = A_i{}'\Gamma_{i+1}A_i - A_i{}'\Gamma_{i+1}B_i\,(R_i + B_i{}'\Gamma_{i+1}B_i)^{-1}\,B_i{}'\Gamma_{i+1}A_i + Q_i$$

with (2.34)

$$\Gamma_N = Q_N$$

Applying the optimal feedback control results in the minimal cost

$$J^* = \underline{x}_o{}'\Gamma_o\underline{x}_o \tag{2.35}$$

By assumption the entire state vector $\underline{x}_i$ is not directly available for measurement and therefore the optimal feedback control can not be implemented. The alternative considered here is to use a minimal-order observer to

construct an estimate $\hat{\underline{x}}_i$ of the state $\underline{x}_i$ and apply the suboptimal feedback control

$$\hat{\underline{u}}_i = \Lambda_i \hat{\underline{x}}_i \tag{2.36}$$

It is of interest to determine the effect of the observer upon the control law. Substituting (2.6) into (2.36) and using the fact that $P_i T_i + V_i H_i = I_n$ it is easily verified that the suboptimal control law is given by the expression

4

$$\hat{\underline{u}}_i = \Lambda_i \underline{x}_i + \Lambda_i P_i \left(\prod_{j=o}^{i-1} F_j \right) (\underline{z}_o - T_o \underline{x}_o) \tag{2.37}$$

It is clear from (2.37) that $\hat{\underline{u}}_i$ is the sum of the optimal control plus an additive term due to the incorrect observer initial condition. The obvious conclusion is that introducing an observer in the loop generally results in an increase in cost from that obtained when the optimal control law is implemented. Further, this increase in cost has been shown by Dellon [10] to be of the form

$$J = J^* + \underline{\varepsilon}_o' \psi_o \underline{\varepsilon}_o \tag{2.38}$$

where the positive definite matrix ψ_i satisfies the recursive equation

$$\psi_i = F_i' \psi_{i+1} F_i + P_i' \Lambda_i' (R_i + B_i' \Lambda_{i+1} B_i) \Lambda_i P_i$$

with (2.39)

$$\psi_N = 0$$

To determine the effect of an observer on the stability properties of a closed loop control system in which it is used we assume it is desired to control the linear system (2.1) by the linear feedback law

$$\underline{u}_i = K_i \underline{x}_i \tag{2.40}$$

Presumably K_i will be chosen by the designer such that the closed loop system, defined by

$$\begin{aligned} \underline{x}_{i+1} &= (A_i + B_i K_i)\underline{x}_i \\ \underline{y}_i &= H_i \underline{x}_i \end{aligned} \tag{2.41}$$

achieves some desirable response properties, which will always include stability. However, the actual state vector $\underline{x}_i$ is not directly available and a discrete time-varying minimal-order observer is used to generate an estimate $\hat{\underline{x}}_i$ of the state $\underline{x}_i$. The estimate $\hat{\underline{x}}_i$ is seen from (2.6) to be of the form

$$\hat{\underline{x}}_i = P_i \underline{z}_i + V_i \underline{y}_i \tag{2.42}$$

where $\underline{z}_i$ is the output of the minimal-order observer and $\underline{y}_i$ is the plant output vector. Applying the control law (2.40) with the state estimate $\hat{\underline{x}}_i$ (2.42) gives the closed-loop state equation

$$\underline{x}_{i+1} = (A_i + B_i K_i V_i H_i)\underline{x}_i + B_i K_i P_i \underline{z}_i \tag{2.43}$$

Also applying the same input to the observer gives

$$\underline{z}_{i+1} = T_{i+1}(A_i + B_i K_i) P_i \underline{z}_i + T_{i+1}(A_i + B_i K_i) V_i H_i \underline{x}_i \tag{2.44}$$

Combining (2.43) and (2.44) results in the following state equation

$$\begin{bmatrix} \underline{x}_{i+1} \\ \hline \underline{z}_{i+1} \end{bmatrix} = \left[\begin{array}{c|c} A_i + B_i K_i V_i H_i & B_i K_i P_i \\ \hline T_{i+1}(A_i + B_i K_i) V_i H_i & T_{i+1}(A_i + B_i K_i) P_i \end{array}\right] \begin{bmatrix} \underline{x}_i \\ \hline \underline{z}_i \end{bmatrix} \tag{2.45}$$

The stability properties of the overall closed-loop system become apparent when the system is viewed in a different state space. With this thought in mind, we perform the coordinate transformation [5]

$$\begin{bmatrix} \underline{x}_i \\ \hline \underline{\varepsilon}_i \end{bmatrix} = \left[\begin{array}{c|c} I_n & 0 \\ \hline -T_i & I_{n-m} \end{array}\right] \begin{bmatrix} \underline{x}_i \\ \hline \underline{z}_i \end{bmatrix} \tag{2.46}$$

This nonsingular transformation results in the equivalent state space representation

$$\begin{bmatrix} \underline{x}_{i+1} \\ \hdashline \underline{\varepsilon}_{i+1} \end{bmatrix} = \left[\begin{array}{c:c} A_i+B_iK_i & -B_iK_iP_i \\ \hdashline 0 & F_i \end{array}\right] \begin{bmatrix} \underline{x}_i \\ \hdashline \underline{\varepsilon}_i \end{bmatrix} \tag{2.47}$$

In the special case of time-invariant systems it is clear that the eigenvalues of the overall system are the eigenvalues of $A + BK$ plus the eigenvalues of the observer system, F. By assumption the closed-loop system $A + BK$ has stable eigenvalues and since the observer is designed to have stable eigenvalues then the overall system is obviously stable. Hence in the time invariant situation it is clear that the observer does not affect the optimal closed-loop poles at all, it merely adds some poles of its own [20]. Intuitively one would expect this same result to carry over to the more general time-varying case. However, although it is true that at any fixed instant "i" the eigenvalues of the system matrix (2.47) are the eigenvalues of $A_i + B_iK_i$ plus the eigenvalues of F_i, this does not imply stability of the overall system (2.47) in any rigorous fashion. To prove stability in the more general case a more careful consideration of the state equations must be taken.

It is, of course, assumed that the designer has constructed a stable time-varying observer. Hence the observer-error is bounded and to prove boundedness of the closed-loop state vector (2.47) it is sufficient to prove boundedness of the subvector $\underline{x}_i$. From (2.47) we have

$$\underline{x}_{i+1} = (A_i + B_i K_i)\underline{x}_i - B_i K_i P_i \underline{\epsilon}_i \tag{2.48}$$

which has the solution

$$\underline{x}_i = \varphi_{i,o}\underline{x}_o - \sum_{j=o}^{i-1} \varphi_{i,j+1} B_j K_j P_j \underline{\epsilon}_j \tag{2.49}$$

where

$$\underline{\epsilon}_i = \left(\prod_{j=o}^{i-1} F_j \right) (\underline{z}_o - T_o \underline{x}_o) \tag{2.50}$$

and the transition matrix $\varphi_{i,j}$ is defined as

$$\varphi_{i,j} \triangleq \prod_{k=j}^{i-1} (A_k + B_k K_k) \tag{2.51}$$

Taking the norm of (2.49)

$$\|\underline{x}_i\| \le \|\varphi_{i,o}\| \, \|\underline{x}_o\| + \sum_{j=o}^{i-1} \|\varphi_{i,j+1}\| \, \|B_j K_j P_j\| \, \|F_{j,o}\| \, \|\underline{\epsilon}_o\| \tag{2.52}$$

Since by assumption $\varphi_{i,o}$ and $F_{i,o}$ are uniformly asymptotically stable we have [10]

$$\|\varphi_{i,o}\| \le c_1 \beta_1^i \quad \text{for some } c_1 > 0 \text{ and } 0 < \beta_1 < 1 \tag{2.53}$$

and

$$\|F_{i,o}\| \le c_2 \beta_2^i \quad \text{for some } c_2 > 0 \text{ and } 0 < \beta_2 < 1 \tag{2.54}$$

Let $\|B_i K_i P_i\| \leq c_3 < \infty$ and (2.52) becomes

$$\|\underline{x}_i\| \leq c_1 \beta_1^i \|\underline{x}_o\| + c_1 c_2 c_3 \sum_{j=0}^{i-1} \beta_1^{i-j-1} \beta_2^j \|\underline{\varepsilon}_o\| \tag{2.55}$$

Evaluating the sum in (2.55) gives

$$\|\underline{x}_i\| \leq c_1 \beta_1^i \|\underline{x}_o\| + c_1 c_2 c_3 \left(\frac{\beta_1^i - \beta_2^i}{\beta_1 - \beta_2} \right) \|\underline{\varepsilon}_o\| \tag{2.56}$$

Thus $\|\underline{x}_i\|$ is bounded for all "i" and since $\lim \|\underline{x}_i\| \to 0$ for all finite $\underline{\varepsilon}_o$ then the closed-loop system (2.45) is uniformly asymptotically stable.

2.4 ADDITIONAL COMMENTS

It should be emphasized at this time that the design procedures of Huddle, Dellon and Williams involve little more than the statement that the designer is free to choose the observer eigenvalues in any desired fashion. The fundamentally important problem of where to place the observer eigenvalues has not yet been solved and remains a perplexing problem to the designer. It is, of course, useful to know that one may design (n-m)-dimensional observers or n-dimensional observers with arbitrary eigenvalues at each instant "i"; however, without the added information of where to optimally place the eigenvalues, the design of the observer remains at best an ad hoc procedure.

In contrast to the idea of artifically picking the observer eigenvalues to provide acceptable system performance, we shall base our observer design on the more fundamental objective of minimizing the effects of system noise disturbances upon the observer derived estimate $\hat{\underline{x}}_i$. In formulating the observer design problem in a more general stochastic setting, the resultant

observer errors will be dependent upon the plant noise disturbances and from a consideration of the noise induced errors an optimal observer design will be obtained. We shall obtain a solution for the observer matrices F_i, T_i and D_i which not only satisfies the fundamental observer equation $T_{i+1}A_i = F_iT_i + D_iH_i$, but results in an observer system which is also optimal in the mean-square sense.

3. OBSERVERS FOR DISCRETE TIME-VARYING SYSTEMS WITH WHITE NOISE INPUTS

3.1 INTRODUCTION

In this chapter we shall focus our attention upon linear, discrete-time stochastic systems for which the dynamic behavior can be characterized by the following set of equations.

$$\underline{x}_{i+1} = A_i\underline{x}_i + B_i\underline{u}_i + \underline{w}_i \tag{3.1}$$

$$\underline{y}_i = H_i\underline{x}_i + \underline{v}_i \tag{3.2}$$

where $\underline{x}_i$ is the n-dimensional state of the system at time "i", $\underline{u}_i$ is the p-dimensional known control vector which acts upon the system at time "i", and $\underline{y}_i$ is the m-dimensional measurement vector. The initial state $\underline{x}_o$ is a Gaussian random vector with known mean and covariance

$$E\{\underline{x}_o\} = \bar{\underline{x}}_o$$

$$E\{(\underline{x}_o-\bar{\underline{x}}_o)(\underline{x}_o-\bar{\underline{x}}_o)'\} = M_o$$

Further, the noise sequences $\underline{w}_i$ and $\underline{v}_i$ are assumed to be Gaussian random vectors with known means and covariances

$$E\{\underline{w}_i\} = \underline{0} \quad \text{for all "i"}$$

$$E\{\underline{v}_i\} = \underline{0} \quad \text{for all "i"}$$

$$E\{\underline{w}_i \underline{w}_j{}'\} = Q_i \delta_{ij}$$

$$E\{\underline{v}_i \underline{v}_j{}'\} = R_i \delta_{ij}$$

where δ_{ij} is the Kronecker delta. In general, the covariance R_i will be considered to be positive definite whereas the covariance Q_i will be positive semi-definite. The various random vectors are also assumed to be mutually uncorrelated so we have the relations

$$E\{\underline{x}_o \underline{w}_i{}'\} = 0 \quad \text{for all "i"}$$

$$E\{\underline{x}_o \underline{v}_i{}'\} = 0 \quad \text{for all "i"}$$

$$E\{\underline{w}_i \underline{v}_j{}'\} = 0 \quad \text{for all "i, j"}$$

Thus it is assumed in this chapter that the noise sequences $\underline{w}_i$ and $\underline{v}_i$ are time-wise uncorrelated sequences which shall be referred to as Gaussian white sequences. In the interest of simplicity, at this point we have assumed a model for the white noise sequences in which the cross-covariance matrix of $\underline{w}_i$ and $\underline{v}_i$ is zero. Later in this chapter we shall extend our results to include the special case whereby $\underline{w}_i$ and $\underline{v}_i$ are Gaussian white sequences which are crosscorrelated at time "i." Also, in the next chapter we shall consider the more general situation in which the noise sequences $\underline{w}_i$ and $\underline{v}_i$ are time-wise correlated sequences of the Gauss-Markov type.

3.2 DEFINITION OF THE DISCRETE OBSERVER FOR STOCHASTIC SYSTEMS

Loosely speaking, for stochastic systems an observer is defined to be a system whose output vector, $\underline{z}_{i+1}$, is an estimate of the quantity $T_{i+1}\underline{x}_{i+1}$ with an estimation error, $\underline{\varepsilon}_{i+1}$, depending only on the previous estimation error, $\underline{\varepsilon}_i$, and the plant and measurement noises $\underline{w}_i, \underline{v}_i$. To be more precise

the discrete time-varying system

$$\underline{z}_{i+1} = F_i\underline{z}_i + G_i\underline{u}_i + D_i\underline{y}_i \tag{3.6}$$

is called an observer of the state $\underline{x}_i$ of the system

$$\underline{x}_{i+1} = A_i\underline{x}_i + B_i\underline{u}_i + \underline{w}_i \tag{3.7}$$

$$\underline{y}_i = H_i\underline{x}_i + \underline{v}_i \tag{3.8}$$

if at each instant "i" the following relation holds

$$\underline{z}_i = T_i\underline{x}_i + \underline{\varepsilon}_i \tag{3.9}$$

where the observer estimation error, $\underline{\varepsilon}_i$, evolves according to the recursive equation

$$\underline{\varepsilon}_{i+1} = F_i\underline{\varepsilon}_i + D_i\underline{v}_i - T_{i+1}\underline{w}_i \tag{3.10}$$

In order that the above relations hold, it is both necessary and sufficient that the following matrix equations be satisfied at each instant "i"

$$T_{i+1}A_i = F_iT_i + D_iH_i \tag{3.11}$$

$$G_i = T_{i+1}B_i \tag{3.12}$$

Necessity is proved as follows. Assuming (3.6) and (3.9) to hold, one obtains the result

$$\begin{aligned}(T_{i+1}A_i - F_iT_i - D_iH_i)\underline{x}_i + (T_{i+1}B_i - G_i)\underline{u}_i \\ + \underline{\varepsilon}_{i+1} - F_i\underline{\varepsilon}_i - D_i\underline{v}_i + T_{i+1}\underline{w}_i = \underline{0}\end{aligned} \tag{3.13}$$

Since (3.13) must be satisfied for all state vectors $\underline{x}_i$ and for all control vectors $\underline{u}_i$, take $\underline{x}_i = \underline{0}$ and $\underline{u}_i = \underline{0}$. This implies the following result.

$$\underline{\varepsilon}_{i+1} = F_i\underline{\varepsilon}_i + D_i\underline{v}_i - T_{i+1}\underline{w}_i \tag{3.14}$$

Hence, (3.13) reduces to the following

$$(T_{i+1}A_i - F_iT_i - D_iH_i)\,\underline{x}_i + (T_{i+1}B_i - G_i)\,\underline{u}_i = \underline{0} \tag{3.15}$$

But (3.15) must hold for all state vectors $\underline{x}_i$ and for all control vectors $\underline{u}_i$, so take $\underline{u}_i = \underline{0}$ and $\underline{x}_i$ arbitrary. This implies the following result.

$$T_{i+1}A_i = F_iT_i + D_iH_i \tag{3.16}$$

Also in (3.15) we may take $\underline{x}_i = \underline{0}$ and $\underline{u}_i$ arbitrary. This implies the following result.

$$G_i = T_{i+1}B_i \tag{3.17}$$

Conversely, assume equations (3.11), (3.12) to be satisfied at each instant "i". Then from (3.6), (3.7) and (3.8) we obtain the following.

$$\begin{aligned}\underline{z}_{i+1} - T_{i+1}\underline{x}_{i+1} &= F_i\underline{z}_i + G_i\underline{u}_i + D_i(H_i\underline{x}_i + \underline{v}_i) \\ &\quad - T_{i+1}(A_i\underline{x}_i + B_i\underline{u}_i + \underline{w}_i)\end{aligned} \tag{3.18}$$

Substituting (3.12) into (3.18) gives the following result.

$$\underline{z}_{i+1} - T_{i+1}\underline{x}_{i+1} = F_i\underline{z}_i + (D_iH_i - T_{i+1}A_i)\,\underline{x}_i + D_i\underline{v}_i - T_{i+1}\underline{w}_i \tag{3.19}$$

Next, since $F_iT_i = -(D_iH_i - T_{i+1}A_i)$ from (3.11), we obtain the result

$$\underline{z}_{i+1} - T_{i+1}\underline{x}_{i+1} = F_i(\underline{z}_i - T_i\underline{x}_i) + D_i\underline{v}_i - T_{i+1}\underline{w}_i \tag{3.20}$$

Clearly, (3.20) implies the following relations

$$\underline{z}_i = T_i\underline{x}_i + \underline{\varepsilon}_i \tag{3.21}$$

$$\underline{\epsilon}_{i+1} = F_i\underline{\epsilon}_i + D_i\underline{v}_i - T_{i+1}\underline{w}_i \tag{3.22}$$

From (3.22) it is seen that the observer error $\underline{\epsilon}_{i+1}$ at time "i+1" depends only upon the previous observer error $\underline{\epsilon}_i$ at time "i" and also on the noise disturbances $\underline{w}_i, \underline{v}_i$.

3.3 AN OPTIMAL MINIMAL-ORDER OBSERVER DESIGN

The discrete time-varying system described by the equations

$$\underline{z}_{i+1} = F_i\underline{z}_i + G_i\underline{u}_i + D_i\underline{y}_i \tag{3.23}$$

$$\underline{z}_i = T_i\underline{x}_i + \underline{\epsilon}_i \tag{3.24}$$

where $\underline{z}_i$, an (n-m)-dimensional vector, is called a minimal-order observer of the state $\underline{x}_i$ of the system (3.1), (3.2) if at each instant "i" the following matrix relations are satisfied

$$T_{i+1}A_i = F_iT_i + D_iH_i \tag{3.25}$$

$$G_i = T_{i+1}B_i \tag{3.26}$$

$$\left[\frac{T_i}{H_i}\right]^{-1} \text{ exists} \tag{3.27}$$

Equation (3.25) is the fundamental observer equation relating the observer system matrices F_i and D_i to the observer transformation matrix T_i. In the design of a minimal-order observer the additional constraint (3.27) must also be satisfied at each instant "i". Using this fact, a general solution to the observer equation (3.25) may be obtained. Rewriting (3.25) in partitioned form

$$T_{i+1}A_i = \left| F_i \middle| D_i \right| \left[\frac{T_i}{H_i} \right] \tag{3.28}$$

and postulating the existence of the matrix inverse to be of the form

$$\left[\frac{T_i}{H_i} \right]^{-1} = \left| P_i \middle| V_i \right| \tag{3.29}$$

where P_i is an n x (n-m) matrix and V_i is an n x m matrix, one obtains upon multiplying (3.28) from the right by the above inverse, the solution

$$F_i = T_{i+1}A_iP_i \tag{3.30}$$

$$D_i = T_{i+1}A_iV_i \tag{3.31}$$

From (3.30), (3.31) it is seen that the design of the minimal-order observer has been reduced to the selection of the single matrix T_i. This is seen from (3.29). Specification of the matrix T_i, together with the known measurement matrix H_i uniquely defines the matrices P_i and V_i and from equations (3.30) and (3.31) is seen to uniquely define the observer system matrices F_i and D_i.

The observer error, $\underline{\varepsilon}_i$, was shown previously to satisfy the following difference equation.

$$\underline{\varepsilon}_{i+1} = F_i\underline{\varepsilon}_i + D_i\underline{v}_i - T_{i+1}\underline{w}_i \tag{3.32}$$

Using the solution (3.30), (3.31) together with the error difference equation (3.32) one obtains the observer error covariance given as (3.33).

$$\overline{\underline{\varepsilon}_{i+1}\underline{\varepsilon}_{i+1}'} = T_{i+1}\,(A_iP_i\overline{\underline{\varepsilon}_i\underline{\varepsilon}_i'}P_i'A_i' + A_iV_iR_iV_i'A_i' + Q_i)\,T_{i+1}' \tag{3.33}$$

We shall define the matrix Ω_i to be the following.

$$\Omega_i \triangleq A_i P_i \overline{\underline{\epsilon}_i \underline{\epsilon}_i{}'} P_i{}' A_i{}' + A_i V_i R_i V_i{}' A_i{}' + Q_i \tag{3.34}$$

It will be useful at this point to partition (3.33) as follows.

$$\overline{\underline{\epsilon}_{i+1} \underline{\epsilon}_{i+1}{}'} = T_{i+1} \left[\begin{array}{c|c} \Omega^i_{11} & \Omega^i_{12} \\ \hline \Omega^i_{21} & \Omega^i_{22} \end{array}\right] T_{i+1}{}' \tag{3.35}$$

where Ω^i_{11} is m x m, Ω^i_{22} is (n-m) x (n-m) and $\Omega^i_{12} = \Omega^i_{21}{}'$ is m x (n-m). The submatrices Ω^i_{11}, Ω^i_{22} and Ω^i_{12} are obtained as partitions of the matrix Ω_i defined by (3.34).

Equation (3.35) plays a fundamental role in the optimal observer design to be developed. We shall next obtain the covariance matrix of the overall estimation error. The estimate $\hat{\underline{x}}_{i+1}$ of the state vector $\underline{x}_{i+1}$ is obtained as follows. (The notation $\hat{\underline{x}}_{i+1}$ shall be used to distinguish between the optimal Kalman filter estimate and the observer derived estimate.) Combining the observer output $\underline{z}_{i+1}$ with the measurement $\underline{y}_{i+1}$ gives the following.

$$\left[\begin{array}{c} \underline{z}_{i+1} \\ \hline \underline{y}_{i+1} \end{array}\right] = \left[\begin{array}{c} T_{i+1} \\ \hline H_{i+1} \end{array}\right] \underline{x}_{i+1} + \left[\begin{array}{c} \underline{\epsilon}_{i+1} \\ \hline \underline{v}_{i+1} \end{array}\right] \tag{3.36}$$

Using the matrix inverse postulated as equation (3.29), we obtain the estimate $\hat{\underline{x}}_{i+1}$.

$$\hat{\underline{x}}_{i+1} = \underline{x}_{i+1} + \left[P_{i+1} \mid V_{i+1}\right] \left[\begin{array}{c} \underline{\epsilon}_{i+1} \\ \hline \underline{v}_{i+1} \end{array}\right] \tag{3.37}$$

The resulting estimation error is found to be

$$\underline{e}_{i+1} \triangleq \hat{\underline{x}}_{i+1} - \underline{x}_{i+1} = \left[P_{i+1} \mid V_{i+1}\right] \left[\begin{array}{c} \underline{\varepsilon}_{i+1} \\ \hline \underline{v}_{i+1} \end{array}\right] \tag{3.38}$$

Finally, the error covariance $\overline{\underline{e}_{i+1}\underline{e}_{i+1}'}$ may be obtained as follows.

$$\overline{\underline{e}_{i+1}\underline{e}_{i+1}'} = \left[P_{i+1} \mid V_{i+1}\right] \left[\begin{array}{c|c} \overline{\underline{\varepsilon}_{i+1}\underline{\varepsilon}_{i+1}'} & 0 \\ \hline 0 & R_{i+1} \end{array}\right] \left[P_{i+1} \mid V_{i+1}\right]' \tag{3.39}$$

where from (3.32) it may be shown that $\overline{\underline{\varepsilon}_{i+1}\underline{v}_{i+1}'} = 0$.

To proceed further, some necessary assumptions must be made about the form of system (3.1), (3.2). It is, of course, assumed that the measurement matrix be of maximal rank at each instant "i" in the interval of interest. In the absence of measurement noise, if H_i did not exhibit this characteristic, then some of the measurements would be linearly dependent and, hence, redundant, so that the measurement vector could be reduced to a linearly independent set without any loss of information. In cases where the system outputs are corrupted by measurement noise, there may however be important reasons to consider all the system outputs, including any redundant ones. We shall not, however, treat this case but shall consider only matrices H_i of full rank.

More specifically, it is assumed that the first "m" columns of H_i (with a possible renumbering of the states) are linearly independent for all "i" in the interval of interest. This is a reasonable assumption in view of the fact that usually the system outputs are affected by the same state variables even though the gains involved may vary with time. In many physical systems H_i

will actually be a constant matrix even though the matrices A_i, B_i are time-varying. Therefore H_i may be partitioned as follows.

$$H_i = \left[H_i^{(1)} | H_i^{(2)}\right] \tag{3.40}$$

where $H_i^{(1)}$ is nonsingular at each instant "i" in the interval of interest. Next we shall assume that $H_i^{(2)}$ is identically zero since the linear transformation

$$\underline{x}_i = \left[\begin{array}{c|c} H_i^{(1)^{-1}} & -H_i^{(1)^{-1}} H_i^{(2)} \\ \hline 0 & I_{n-m} \end{array}\right] \underline{q}_i \tag{3.41}$$

will transform the original system to the desired form shown in equation (3.42). Therefore, without loss of generality, it will be assumed that the measurements are of the form

$$\underline{y}_i = [I_m \mid 0\,]\underline{x}_i + \underline{v}_i \tag{3.42}$$

To complete the basic observer design, it remains only to specify the observer matrix T_i. Since the matrix $\left[\frac{T_i}{H_i}\right]^{-1}$ must exist at each instant "i", the most logical choice for the matrix T_i is given below as (3.43).

$$T_i = [K_i \mid I_{n-m}\,] \tag{3.43}$$

K_i is an arbitrary (n-m) x m gain matrix which will be chosen to minimize the overall estimation error. With this choice for the matrix T_i, the matrices P_i and V_i are found to be the following

$$P_i = \left[\frac{0}{I_{n-m}}\right], \qquad V_i = \left[\frac{I_m}{-K_i}\right] \tag{3.44}$$

Substituting (3.44) into (3.39) gives the following result.

$$\overline{\underline{e}_{i+1}\underline{e}_{i+1}'} = \left[\begin{array}{c|c} R_{i+1} & -R_{i+1}K_{i+1}' \\ \hline -K_{i+1}R_{i+1} & \overline{\underline{\epsilon}_{i+1}\underline{\epsilon}_{i+1}'} + K_{i+1}R_{i+1}K_{i+1}' \end{array}\right] \tag{3.45}$$

Also, substituting (3.43) into (3.35) gives the result

$$\overline{\underline{\epsilon}_{i+1}\underline{\epsilon}_{i+1}'} = K_{i+1}\Omega^i_{11}K_{i+1}' + K_{i+1}\Omega^i_{12} + \Omega^i_{21}K_{i+1}' + \Omega^i_{22} \tag{3.46}$$

The optimal gain matrix K_{i+1}^* may now be determined. From (3.45) and (3.46) one obtains

$$\begin{aligned} \text{trace}\,\overline{\underline{e}_{i+1}\underline{e}_{i+1}'} &= \text{trace}\, R_{i+1} \\ &+ \text{trace}\,\{K_{i+1}(\Omega^i_{11} + R_{i+1})K_{i+1}' + K_{i+1}\Omega^i_{12} + \Omega^i_{21}K_{i+1}' + \Omega^i_{22}\} \end{aligned} \tag{3.47}$$

"Completing the square" in (3.47) gives the result

$$\begin{aligned} \text{trace}\,\overline{\underline{e}_{i+1}\underline{e}_{i+1}'} &= \text{trace}\, R_{i+1} \\ &+ \text{trace}\,\{[K_{i+1} + \Omega^i_{21}(\Omega^i_{11} + R_{i+1})^{-1}][\Omega^i_{11} + R_{i+1}][K_{i+1} + \\ &+ \Omega^i_{21}(\Omega_{11} + R_{i+1})^{-1}]' + \Omega^i_{22} - \Omega^i_{21}(\Omega^i_{11} + R_{i+1})^{-1}\Omega^i_{12}\} \end{aligned} \tag{3.48}$$

The desired optimal gain matrix, K_{i+1}^*, is obtained by minimizing the trace $\overline{\underline{e}_{i+1}\underline{e}_{i+1}'}$. By assumption the measurement noise covariance R_{i+1} is positive definite. Clearly, the submatrix Ω^i_{11} is at least positive semi-definite so that the matrices $(\Omega^i_{11} + R_{i+1})$ and $(\Omega^i_{11} + R_{i+1})^{-1}$ are positive definite. Therefore the matrices

$$[K_{i+1} + \Omega^i_{21}(\Omega^i_{11} + R_{i+1})^{-1}][\Omega^i_{11} + R_{i+1}][K_{i+1}+\Omega^i_{21}(\Omega^i_{11} + R_{i+1})^{-1}]' \tag{3.49}$$

and

$$\Omega^i_{21}(\Omega^i_{11} + R_{i+1})^{-1}\Omega^i_{12} \tag{3.50}$$

must have positive diagonal elements. The minimum of the diagonal elements of $\overline{\underline{e}_{i+1}\underline{e}_{i+1}'}$ must therefore be attained when

$$K_{i+1} + \Omega^i_{21} (\Omega^i_{11} + R_{i+1})^{-1} = 0 \tag{3.51}$$

Clearly the optimal gain matrix K^*_{i+1} is given by the expression

$$K_{i+1}{}^* = -\Omega^i_{21}(\Omega^i_{11} + R_{i+1})^{-1} \tag{3.52}$$

The minimal estimation error obtained when $K_{i+1}{}^*$ is taken to be the observer gain matrix is found by substituting equation (2.39) into equation (2.35). Thus it is found that

$$\min \text{trace } \overline{\underline{e}_{i+1}\underline{e}_{i+1}'} = \text{trace } R_{i+1} + \text{trace } [\Omega^i_{22} - \Omega^i_{21}(\Omega^i_{11}+R_{i+1})^{-1}\Omega^i_{12}] \tag{3.53}$$

Design of the optimal minimal-order observer is essentially complete at this point; it remains only to specify the resulting observer dynamical structure.

Previously it was shown that the observer matrices were of the form $F_i = T_{i+1}A_iP_i$ and $D_i = T_{i+1}A_iV_i$. Straightforward substitution of the observer transformation matrix T_{i+1} [equation (3.43)] and the corresponding matrices P_i and V_i [equation (3.44)] results in the following

$$F_i = A_{22}^i + K_{i+1}A_{12}^i \tag{3.54}$$

$$D_i = A_{21}^i - A_{22}^i K_i + K_{i+1}(A_{11}^i - A_{12}^i K_i) \tag{3.55}$$

Also, the matrix G_i is defined explicitly in terms of the observer transformation T_{i+1} and the plant matrix B_i according to the relation

$$G_i = T_{i+1}B_i \tag{3.56}$$

A block diagram of the basic observer structure is shown in Figure 3.1 along with the appropriate defining equations and the algorithm for obtaining the optimal observer gain matrix.

3.4 INITIALIZATION OF THE DISCRETE OBSERVER

In the case of the recursive Kalman filter equations, the a priori statistics $\bar{\underline{x}}_o$ and M_o of the initial state $\underline{x}_o$ are assumed to be known. This a priori information is needed to initialize the Kalman filter. Since the optimal observer equations (3.35) and (3.52) are also recursive, this same information is needed to initialize the observer. We shall therefore assume that the a priori statistics $\bar{\underline{x}}_o$ and M_o are available to the observer system.

Initialization of the observer proceeds as follows. Let $\underline{z}_1 = T_1\bar{\underline{x}}_1$ be the observer initial condition, where $\bar{\underline{x}}_1$ is the "expected value" of the state vector $\underline{x}_1$. Since $\underline{\varepsilon}_1 = \underline{z}_1 - T_1\underline{x}_1$, then

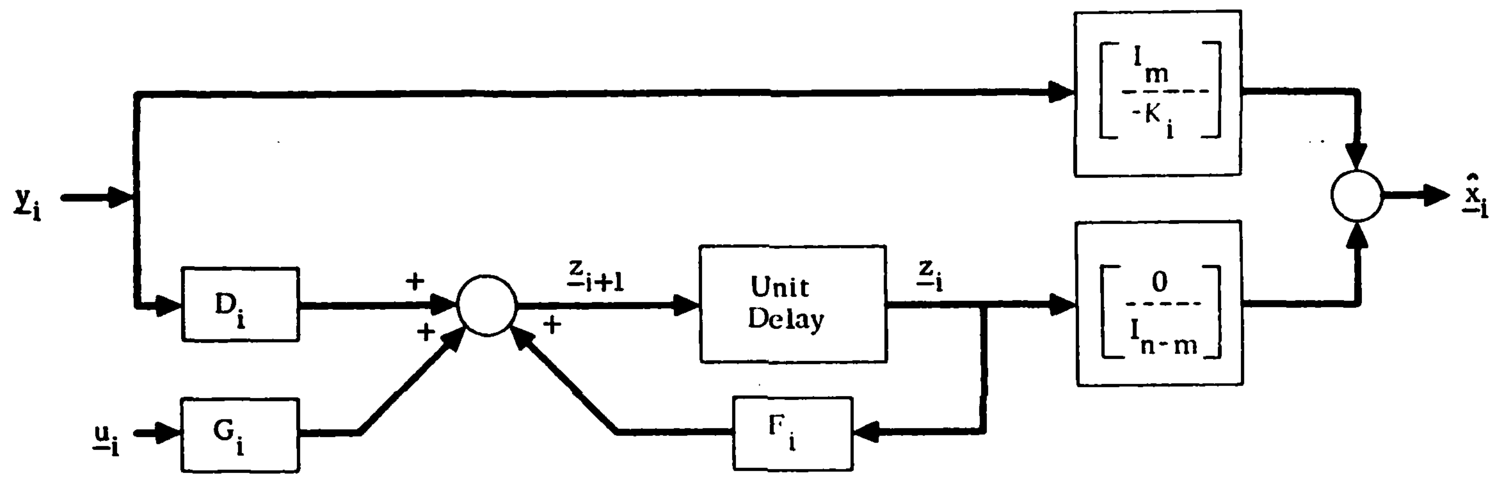
$$\overline{\underline{\varepsilon}_1\underline{\varepsilon}_1'} = T_1\overline{(\underline{x}_1 - \bar{\underline{x}}_1)(\underline{x}_1 - \bar{\underline{x}}_1)'}\,T_1' \tag{3.57}$$

But $\underline{x}_1 - \bar{\underline{x}}_1 = A_o(\underline{x}_o - \bar{\underline{x}}_o) + \underline{w}_o$ hence (3.57) becomes

$$\overline{\underline{\varepsilon}_1\underline{\varepsilon}_1'} = T_1(A_oM_oA_o' + Q_o)T_1' \tag{3.58}$$

To initialize the observer, define the covariance matrix Ω_o to be

$$\Omega_o = A_oM_oA_o' + Q_o \tag{3.59}$$



Observer Structure

$F_i = A_{22}{}^i + K_{i+1} A_{12}{}^i$

$D_i = A_{21}{}^i - A_{22}{}^i K_i + K_{i+1}(A_{11}{}^i - A_{12}{}^i K_i)$

$G_i = T_{i+1} B_i$

$P_i = \begin{bmatrix} 0 \\ I_{n-m} \end{bmatrix}$, $V_i = \begin{bmatrix} I_m \\ -K_i \end{bmatrix}$

$T_i = [K_i \mid I_{n-m}]$

Optimal Gain Algorithm

$K_{i+1} = -\Omega_{21}{}^i (\Omega_{11}{}^i + R_{i+1})^{-1}$

$\Omega_i = A_i P_i \overline{\underline{\epsilon}_i \underline{\epsilon}_i{}'} P_i{}' A_i{}' + A_i V_i R_i V_i{}' A_i{}' + Q_i$

$\Omega_o = A_o M_o A_o{}' + Q_o$

$\overline{\underline{\epsilon}_{i+1} \underline{\epsilon}_{i+1}{}'} = T_{i+1} \Omega_i T_{i+1}{}'$

Figure 3.1 Optimal Minimal-Order Observer Structure.

and take the optimal gain matrix K_1^* to be

$$K_1^* = -\Omega_{21}^o(\Omega_{11}^o + R_1)^{-1} \tag{3.60}$$

3.5 SPECIAL CASE: CROSS CORRELATED PLANT AND MEASUREMENT NOISES

In the interest of simplicity we have assumed that the original model of the Gaussian white noise sequences is one in which the cross-covariance matrix of $\underline{w}_i$ and $\underline{v}_i$ is zero. We shall now treat this important special case in which the cross-covariance matrix of $\underline{w}_i$ and $\underline{v}_i$ is non-zero and we shall show that the observer design technique described in the previous sections of this chapter is directly applicable to this special case with only minor modifications to the theory. At this point we shall assume that the zero-mean Gaussian white sequences $\underline{w}_i$ and $\underline{v}_i$ are characterized by the covariance relations:

$$E\{\underline{w}_i\underline{w}_j'\} = Q_i\delta_{ij}$$

$$E\{\underline{v}_i\underline{v}_j'\} = R_i\delta_{ij}$$

$$E\{\underline{w}_i\underline{v}_j'\} = S_i\delta_{ij}$$

We begin by computing the observer-error covariance matrix. From the basic observer-error equation (3.10) we obtain the result

$$\begin{aligned}\overline{\underline{\epsilon}_{i+1}\underline{\epsilon}_{i+1}'} = F_i\overline{\underline{\epsilon}_i\underline{\epsilon}_i'}F_i' + D_i\overline{\underline{v}_i\underline{v}_i'}D_i' + T_{i+1}\overline{\underline{w}_i\underline{w}_i'}T_{i+1}' \\ - D_i\overline{\underline{v}_i\underline{w}_i'}T_{i+1}' - T_{i+1}\overline{\underline{w}_i\underline{v}_i'}D_i'\end{aligned} \tag{3.61}$$

But since $F_i = T_{i+1}A_iP_i$ and $D_i = T_{i+1}A_iV_i$, substituting these relations into (3.61) gives

$$\overline{\underline{\varepsilon}_{i+1}\underline{\varepsilon}_{i+1}'} = T_{i+1}(A_iP_i\,\overline{\underline{\varepsilon}_i\underline{\varepsilon}_i'}\,P_i'A_i' + A_iV_iR_iV_i'A_i' + Q_i)\,T_{i+1}'$$
$$-T_{i+1}(A_iV_iS_i + S_i'V_i'A_i')\,T_{i+1}' \tag{3.62}$$

Hence, the same general form of solution is obtained as in the previous uncrosscorrelated noise case. Defining the matrix Ω_i to be

$$\Omega_i = A_iP_i\overline{\underline{\varepsilon}_i\underline{\varepsilon}_i'}\,P_i'A_i' + A_iV_iR_iV_i'A_i' + Q_i$$
$$- A_iV_iS_i - S_i'V_i'A_i' \tag{3.63}$$

and partitioning (3.63) as before we obtain the result

$$\overline{\underline{\varepsilon}_{i+1}\underline{\varepsilon}_{i+1}'} = T_{i+1}\left[\begin{array}{c|c} \Omega^i_{11} & \Omega^i_{12} \\ \hline \Omega^i_{21} & \Omega^i_{22} \end{array}\right] T_{i+1}' \tag{3.64}$$

It is immediately obvious that the observer design developed previously in this chapter applies without modification from this point on. For the sake of brevity we shall state only the final results. Taking the observer transformation matrix T_{i+1} to be of the form

$$T_{i+1} = [K_{i+1} \mid I_{n-m}] \tag{3.65}$$

The optimal gain matrix K_{i+1}^* is found to be

$$K_{i+1}^* = -\Omega^i_{21}(\Omega^i_{11} + R_{i+1})^{-1} \tag{3.66}$$

where the matrices Ω^i_{11} and Ω^i_{21} are obtained from the partitioned Ω_i matrix as indicated in (3.64). The cross-covariance matrix, S_i, alters only the computation of the Ω_i matrix as indicated in (3.63).

3.6 EFFECT OF COORDINATE TRANSFORMATION ON OPTIMAL GAIN MATRIX, K_i^*

Up to this point it has been tacitly assumed that "without loss of generality" the given system (3.1), (3.2) was already in the desired canonical form. However, the phrase "without loss of generality" needs to be justified since for many dynamical systems the desired canonical form cannot be obtained directly by merely renumbering the state variables. We do, however, assume that the system measurement matrix, H_i, can be put into the form (3.40) and then the linear transformation (3.41) applied to obtain the desired canonical equations. If this transformation need be used, then there will be a modification to the optimal gain matrix, K_{i+1}^*, due to the linear transformation (3.41). We shall now consider the effect of this linear transformation upon our optimization technique and, in particular, we shall derive the optimal gain matrix taking into account the effect of the linear transformation (3.41).

Assume it is necessary to apply the linear transformation $\underline{x}_i = M_i \underline{q}_i$, where M_i is defined in (3.41). Upon performing this transformation we have the measurements

$$\underline{y}_i = [I_m \mid 0\,]\underline{q}_i + \underline{v}_i \tag{3.67}$$

Let the observer be defined by the system (3.6) where now we take the observer output to be

$$\underline{z}_i = [\,K_i \mid I\,]\underline{q}_i + \underline{\varepsilon}_i \tag{3.68}$$

Combining (3.67) and (3.68) together with the fact that $\underline{x}_i = M_i \underline{q}_i$ we get the result

$$\begin{bmatrix} \underline{z}_i \\ \hline \underline{y}_i \end{bmatrix} = \left[\begin{array}{c|c} K_i & I_{n-m} \\ \hline I_m & 0 \end{array}\right] \left[\begin{array}{c|c} H_i^{(1)} & H_i^{(2)} \\ \hline 0 & I_{n-m} \end{array}\right] \underline{x}_i + \begin{bmatrix} \underline{\epsilon}_i \\ \hline \underline{v}_i \end{bmatrix} \tag{3.69}$$

From (3.69) the overall estimation error covariance is found to be the following:

$$\overline{\underline{e}_i\underline{e}_i{}'} = \left[\begin{array}{c|c} H_i^{(1)^{-1}} & -H_i^{(1)^{-1}} H_i^{(2)} \\ \hline 0 & I_{n-m} \end{array}\right] \left[\begin{array}{c|c} R_i & -R_iK_i{}' \\ \hline -K_iR_i & \overline{\underline{\epsilon}_i\underline{\epsilon}_i{}'} + K_iR_iK_i{}' \end{array}\right] \left[\begin{array}{c|c} H_i^{(1)^{-1}} & -H_i^{(1)^{-1}} H_i^{(2)} \\ \hline 0 & I_{n-m} \end{array}\right]' \tag{3.70}$$

Equation (3.70) is simply the statement that the error in the $\underline{x}_i$ coordinate system is M_i times the error in the $\underline{q}_i$ coordinate system. That is,

$$(\underline{x}_i - \hat{\underline{x}}_i) = M_i(\underline{q}_i - \hat{\underline{q}}_i) \tag{3.71}$$

Performing the matrix multiplication indicated in (3.70) and taking the trace gives the result

$$\begin{aligned} \text{trace } \overline{\underline{e}_i\underline{e}_i{}'} &= \text{trace } \{\overline{\underline{\epsilon}_i\underline{\epsilon}_i{}'} + K_iR_iK_i{}'\} \\ &+ \text{trace } \Big\{ \left(H_i^{(1)^{-1}} H_i^{(2)}\right) \overline{\underline{\epsilon}_i\underline{\epsilon}_i{}'} \left(H_i^{(1)^{-1}} H_i^{(2)}\right)' \\ &+ \left(H_i^{(1)^{-1}} + H_i^{(1)^{-1}} H_i^{(2)} K_i\right) R_i \left(H_i^{(1)^{-1}} + H_i^{(1)^{-1}} H_i^{(2)} K_i\right)' \Big\} \end{aligned} \tag{3.72}$$

where the observer error covariance is of the form

$$\overline{\underline{\epsilon}_i\underline{\epsilon}_i}' = K_i\Omega_{11}^{i-1}K_i^1 + K_i\Omega_{12}^{i-1} + \Omega_{21}^{i-1}K_i^1 + \Omega_{22}^{i-1} \tag{3.73}$$

Setting the gradient of (3.72) (with respect to the free gain matrix, K_i) equal to zero gives the first order necessary conditions for a minimum. Since (3.72) is quadratic in K_i, these first order necessary conditions are also sufficient conditions for a minimum. To obtain the gradient of (3.72) one first substitutes (3.73) into (3.72) and expands the trace $\overline{\underline{e}_i\underline{e}_i}'$ as follows.

$$\begin{aligned}
\text{trace}\ \overline{\underline{e}_i\underline{e}_i}' = {} & \text{tr}\,\{K_i(\Omega_{11}^{i-1} + R_i)K_i'\} + 2\ \text{tr}\,\{\Omega_{21}^{i-1}K_i'\} + \text{tr}\,\{\Omega_{22}^{i-1}\} \\
& + \text{tr}\,\{(\cdot)'(\cdot)\,K_i\Omega_{11}^{i-1}K_i'\} + 2\ \text{tr}\,\{(\cdot)'(\cdot)\,\Omega_{21}^{i-1}K_i'\} \\
& \qquad + \text{tr}\,\{(\cdot)'(\cdot)\,\Omega_{22}^{i-1}\} \\
& + \text{tr}\,\{(\cdot)'(\cdot)K_iR_iK_i'\} + 2\,\text{tr}\left\{(\cdot)H_i^{(1)^{-1}}R_iK_i'\right\} \\
& \qquad + \text{tr}\left\{\left(H_i^{(1)^{-1}}\right)'\left(H_i^{(1)^{-1}}\right)R_i\right\}
\end{aligned} \tag{3.74}$$

where $(\cdot) \triangleq H_i^{(1)^{-1}}H_i^{(2)}$.

Using the formulae given in Athans [6], the gradient of (3.74) is evaluated and set equal to zero giving the result:

$$\begin{aligned}
& \Big(I + (\cdot)'(\cdot)\Big)K_i(\Omega_{11}^{i-1} + R_i) \\
& \qquad = -\Big(I + (\cdot)'(\cdot)\Big)\Omega_{21}^{i-1} - (\cdot)'\,H_i^{(1)^{-1}}R_i
\end{aligned} \tag{3.75}$$

But since the matrices $\left(I + (\cdot)'(\cdot)\right)$ and $\left(\Omega_{11}^{i-1} + R_i\right)$ are positive definite (hence invertible) we obtain from (3.75) the result

$$K_i^* = -(\Omega_{21}^{i-1} + [\,\cdot\,])(\Omega_{11}^{i-1} + R_i)^{-1} \tag{3.76}$$

where

$$[\,\cdot\,] \triangleq \left(I + (\cdot)'(\cdot)\right)(\cdot)' H_i^{(1)^{-1}} R_i \tag{3.77}$$

Finally we note that for the special case where $H_i^{(2)}$ is identically zero, the term $[\,\cdot\,]$ is identically zero and the optimal gain (3.76) reduces to the result (3.52) obtained previously.

3.7 GENERALITY OF THE TRANSFORMATION $T_i = [K_i | I_{n-m}]$

At this point one might ask if the consideration of a more general observer transformation, T_i, could result in a further reduction in mean-square estimation error. To be more specific, can the mean-square estimation error be reduced even further by taking $T_i = [K_i^{(1)} | K_i^{(2)}]$ instead of using the less general transformation $T_i = [K_i^{(1)} | I_{n-m}]$? The answer to this question is an unequivocable "no" and in this section of the thesis we shall present a proof of the claim. The proof is straightforward.

We assume that the measurements are already in the desired canonical form, that is:

$$\underline{y}_i = [I_m | 0]\,\underline{x}_i + \underline{v}_i \tag{3.78}$$

We consider the most general possible observer transformation, T_i, which is of the form

$$\underline{z}_i = [K_i^{(1)} | K_i^{(2)}]\underline{x}_i + \underline{\varepsilon}_i \tag{3.79}$$

where $K_i^{(1)}$ and $K_i^{(2)}$ are (n-m) x m and (n-m) x (n-m) partitions of the matrix T_i. Since the matrix inverse $\left[\frac{H_i}{T_i}\right]^{-1}$ is required to exist at each instant "i" then we have the result

$$\det\left[\begin{array}{c|c} I_m & 0 \\ \hline K_i^{(1)} & K_i^{(2)} \end{array}\right] = \det[K_i^{(2)}] \tag{3.80}$$

and therefore we consider all transformations $T_i = [K_i^{(1)} | K_i^{(2)}]$ where $K_i^{(1)}$ and $K_i^{(2)}$ are arbitrary and $K_i^{(2)}$ is full rank at each "i." We shall now prove that at each step "i" there is no loss of generality by taking $K_i^{(2)} = I_{n-m}$ and this is because the minimum achievable mean-square estimation error is, *in fact*, independent of the elements of the partition $K_i^{(2)}$.

In the first step of the proof we treat the initialization of the observer. Computing the mean-square estimation error at time "i=1" we obtain the result

$$\begin{aligned} \text{trace}\,\overline{\underline{\varepsilon}_1\underline{\varepsilon}_1{}'} &= \text{trace}\,R_1 \\ &+ \text{trace}\left\{K_1^{(2)^{-1}}\,\overline{\underline{\varepsilon}_1\underline{\varepsilon}_1{}'}\,K_1^{(2)^{-1}{}'} + K_1^{(2)^{-1}}K_1^{(1)}R_1K_1^{(1)}{}'K_1^{(2)^{-1}{}'}\right\} \end{aligned} \tag{3.81}$$

But the observer error covariance is $\overline{\underline{\varepsilon}_1\underline{\varepsilon}_1{}'} = T_1\Omega_0T_1{}'$ where $\Omega_0 \triangleq A_0M_0A_0{}' + Q_0$, so expanding $\overline{\underline{\varepsilon}_1\underline{\varepsilon}_1{}'}$ in (3.81) into quadratic terms involving the appropriate partitions of the matrix Ω_0 gives the result (3.82).

$$\overline{\underline{\epsilon}_1\underline{\epsilon}_1}' = K_1^{(1)}\Omega_{11}^{o}K_1^{(1)\prime} + K_1^{(1)}\Omega_{12}^{o}K_1^{(2)\prime} + K_1^{(2)}\Omega_{21}^{o}K_1^{(1)\prime}$$

$$+ K_1^{(2)}\Omega_{22}^{o}K_1^{(2)\prime} \qquad (3.82)$$

Substituting (3.82) into (3.81) and "completing the square" gives the expression

$$\text{trace } \overline{\underline{e}_1\underline{e}_1}' = \text{trace } R_1$$

$$+ \text{trace}\left\{\left[K_1^{(2)^{-1}}K_1^{(1)} + \Omega_{21}^{o}(\Omega_{11}^{o} + R_1)^{-1}\right]\left[\Omega_{11}^{o} + R_1\right]\left[K_1^{(2)^{-1}}K_1^{(1)}\right.\right.$$

$$\left.\left. + \Omega_{21}^{o}(\Omega_{11}^{o} + R_1)^{-1}\right]' + \Omega_{22}^{o} - \Omega_{21}^{o}(\Omega_{11}^{o} + R_1)^{-1}\,\Omega_{21}^{o\prime}\right\} \qquad (3.83)$$

Clearly, to minimize trace $\overline{\underline{e}_1\underline{e}_1}'$ we take

$$K_1^{(2)^{-1}}K_1^{(1)} + \Omega_{21}^{o}(\Omega_{11}^{o} + R_1)^{-1} = 0 \qquad (3.84)$$

and the minimum attainable mean-square error is given by the result

$$\min \text{trace } \overline{e_1e_1}' = \text{trace } R_1 + \text{trace }\{\Omega_{22}^{o} - \Omega_{21}^{o}(\Omega_{11}^{o} + R_1)^{-1}\,\Omega_{21}^{o\prime}\}$$

$$(3.85)$$

We note at this point that the optimal error (3.85) is attained independent of the particular choice of $K_1^{(2)}$. Hence, the minimum attainable mean-square error is independent of the partition $K_1^{(2)}$ and we may without loss of generality take $K_1^{(2)} = I_{n-m}$.

For all cases n=2, 3, ..., i, i+1 the solution proceeds as follows. At time "i+1" the equations of interest are the following:

$$\underline{y}_{i+1} = [I_m | 0]\, \underline{x}_{i+1} + \underline{v}_{i+1}$$

$$\underline{z}_{i+1} = [K_{i+1}^{(1)} | K_{i+1}^{(2)}]\, \underline{x}_{i+1} + \underline{\varepsilon}_{i+1} \tag{3.86}$$

Also,

$$\overline{\underline{\varepsilon}_{i+1}\underline{\varepsilon}_{i+1}{}'} = T_{i+1}\Omega_i T_{i+1}{}'$$

where

$$\Omega_i \triangleq A_i P_i \overline{\underline{\varepsilon}_i \underline{\varepsilon}_i{}'} P_i{}' A_i{}' + A_i V_i R_i V_i{}' A_i + Q_i \tag{3.87}$$

Repeating the procedure described for "i=1" we find the mean-square estimation error at time "i+1" to be

$$\text{trace}\, \overline{\underline{e}_{i+1}\underline{e}_{i+1}{}'} = \text{trace}\, R_{i+1}$$

$$+ \text{trace}\left\{ \left[K_{i+1}^{(2)^{-1}} K_{i+1}^{(1)} + \Omega_{21}^i \left(\Omega_{11}^i + R_{i+1}\right)\right]^{-1} \left[\Omega_{11}^i + R_{i+1}\right] \cdot \right.$$

$$\left[K_{i+1}^{(2)^{-1}} K_{i+1}^{(1)} + \Omega_{21}^i \left(\Omega_{11}^i + R_{i+1}\right)^{-1}\right]'$$

$$\left. + \Omega_{22}^i - \Omega_{21}^i \left(\Omega_{11}^i + R_{i+1}\right)^{-1} \Omega_{21}^i{}' \right\} \tag{3.88}$$

Clearly, to minimize trace $\overline{\underline{e}_{i+1}\underline{e}_{i+1}{}'}$ we take

$$K_{i+1}^{(2)^{-1}} K_{i+1}^{(1)} + \Omega_{21}^i \left(\Omega_{11}^i + R_{i+1}\right)^{-1} = 0 \tag{3.89}$$

The minimum attainable mean-square estimation error at time "i+1" is given by the result

$$\min \operatorname{trace} \overline{\underline{e}_{i+1}\underline{e}_{i+1}'} = \operatorname{trace} R_{i+1}$$

$$+ \operatorname{trace}\left\{\Omega_{22}^{i} - \Omega_{21}^{i}\left(\Omega_{11}^{i} + R_{i+1}\right)^{-1} \Omega_{21}^{i}{}'\right\} \tag{3.90}$$

and this optimal result (3.90) is attained independent of the particular choice of $K_{i+1}^{(2)}$. Therefore, at each step n=2, 3, ..., i, i+1, ..., without loss of generality, we may take $K_{i+1}^{(2)} = I_{n-m}$.

3.8 EQUIVALENCE OF OBSERVER AND KALMAN FILTER WHEN $R_i = 0$

Up to this point it has been a basic assumption that the measurement noise be non-zero and, in fact, it was more strongly assumed that the measurement noise covariance, R_i, be positive definite at each instant "i." This corresponds to the case where each measurement component is contaminated by an independent white noise disturbance. A special case of particular interest is the opposite extreme where the measurements are completely noise-free, that is, $\underline{v}_i = \underline{0}$ for all "i." We shall next treat this important special case.

Rather loosely stated, in the absence of measurement noise, "m" of the system states are known exactly and it is only necessary to estimate the remaining "n-m" states. In this particular situation it is clear that the Kalman filter is degenerate in the sense that it reduces to an "n-m" dimensional filter. Noting that the minimal-order observer is of dimension "n-m," one questions whether or not in this situation (i.e., in the absense of measurement noise) the optimal minimal-order observer is equivalent to the Kalman filter in the sense that both filters provide identical mean-square

estimation errors. We shall demonstrate that this property is, in fact, true.

We assume the system equations are in the form

$$\begin{bmatrix} \underline{x}_{i+1}^{(1)} \\ \hline \underline{x}_{i+1}^{(2)} \end{bmatrix} = \left[\begin{array}{c|c} A_{11}^{i} & A_{12}^{i} \\ \hline A_{21}^{i} & A_{22}^{i} \end{array}\right] \begin{bmatrix} \underline{x}_{i}^{(1)} \\ \hline \underline{x}_{i}^{(2)} \end{bmatrix} + \begin{bmatrix} \underline{w}_{i}^{(1)} \\ \hline \underline{w}_{i}^{(2)} \end{bmatrix} \tag{3.91}$$

$$\underline{y}_i = [I_m \mid 0]\, \underline{x}_i \tag{3.92}$$

For purposes of simplicity the plant noise covariance is assumed to be:

$$Q_i = \left[\begin{array}{c|c} Q_i^{(1)} & 0 \\ \hline 0 & Q_i^{(2)} \end{array}\right] \tag{3.93}$$

Using the Kalman filter algorithms (see Chapter 1, equations (1.3) through (1.6) it is easily verified that for the system defined by (3.91), (3.92) the mean-square error for the Kalman filter is

$$\text{trace}\, P_{i+1/i+1} =$$

$$\text{trace}\Big\{ \left(A_{22}^{i} P_{i/i}^{(2)} A_{22}^{i\,\prime} + Q_i^{(2)}\right)$$

$$- A_{22}^{i} P_{i/i}^{(2)} A_{12}^{i\,\prime} \left(A_{12}^{i} P_{i/i}^{(2)} A_{12}^{i\,\prime} + Q_i^{(1)}\right)^{-1} A_{12}^{i} P_{i/i}^{(2)} A_{22}^{i\,\prime} \Big\} \tag{3.94}$$

where the covariance $P_{i+1/i+1}$ is partitioned in the form

$$P_{i+1/i+1} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & P_{i+1/i+1}^{(2)} \end{array}\right] \tag{3.95}$$

Next, from the observer error covariance $\overline{\underline{\epsilon}_{i+1}\underline{\epsilon}_{i+1}'}$, which is

$$\overline{\underline{\epsilon}_{i+1}\underline{\epsilon}_{i+1}'} = T_{i+1}\left[\begin{array}{c|c} A^i_{12}\overline{\underline{\epsilon}_i\underline{\epsilon}_i'}A^{i\,\prime}_{12} + Q^{(1)}_i & A^i_{12}\overline{\underline{\epsilon}_i\underline{\epsilon}_i'}A^{i\,\prime}_{22} \\ \hline A^i_{22}\overline{\underline{\epsilon}_i\underline{\epsilon}_i'}A^{i\,\prime}_{12} & A^i_{22}\overline{\underline{\epsilon}_i\underline{\epsilon}_i'}A^{i\,\prime}_{22} + Q^{(2)}_i \end{array}\right]T_{i+1}' \tag{3.96}$$

it is found that the optimal observer estimation error is

$$\mathrm{trace}\,\overline{\underline{e}_{i+1}\underline{e}_{i+1}'} =$$

$$\mathrm{trace}\left\{\left(A^i_{22}\overline{\underline{\epsilon}_i\underline{\epsilon}_i'}A^{i\,\prime}_{22} + Q^{(2)}_i\right) - A^i_{22}\overline{\underline{\epsilon}_i\underline{\epsilon}_i'}A^{i\,\prime}_{12}\left(A^i_{12}\overline{\underline{\epsilon}_i\underline{\epsilon}_i'}A^{i\,\prime}_{12} + Q^{(1)}_i\right)^{-1}A^i_{12}\overline{\underline{\epsilon}_i\underline{\epsilon}_i'}A^{i\,\prime}_{22}\right\} \tag{3.97}$$

Equivalence of (3.94) and (3.97) follows directly from the result that in the case of no measurement noise, $\overline{\underline{\epsilon}_i\underline{\epsilon}_i'} = P^{(2)}_{i/i}$. This result is obtained by inspection of the observer estimation error covariance $\overline{\underline{e}_{i+1}\underline{e}_{i+1}'}$ and the corresponding relation $P_{i+1/i+1}$ for the Kalman filter.

4. OBSERVERS FOR DISCRETE SYSTEMS WITH GAUSS-MARKOV NOISE INPUTS

4.1 INTRODUCTION

In the previous chapter we have limited ourselves to estimation problems in which the system disturbances were modeled as purely random additive white sequences. Clearly, in many estimation problems the system noises will be modeled more accurately as additive Gauss-Markov sequences (time-wise correlated noise sequences). Sequentially correlated plant noises can, in principle, be treated by introducing shaping filters driven by purely random white sequences resulting in sequentially correlated sequences.[29] However, in the design of the Kalman filter for systems with sequentially correlated noise inputs it is necessary to increase the dimension of the state vector to be estimated. This is inconvenient for real-time filtering and, equally important, the computation of the Kalman filter gains is very likely to be ill-conditioned. Thus it is desirable to seek better ways to handle sequentially correlated plant disturbances in estimation problems.

4.2 OBSERVER DESIGN FOR SYSTEMS WITH GAUSS-MARKOV PLANT NOISE

We shall now extend the results of the observer theory developed in the previous chapter to the problem of estimation in the presence of time-wise correlated plant disturbances. This problem shall be treated in a straightforward manner, that is, the state equations of the plant will not be augmented as must be done in the Kalman filtering theory. Taking this

direct approach will result in an observer of minimal dimension. To be more precise, the dimension of the minimal-order observer will be the same as for the case when the plant noises are purely random Gaussian white sequences. The order of the observer will therefore be independent of the dimension of the linear system required to generate the Gauss-Markov sequence. The resulting observer is not designed to provide estimates of the extra states which model the plant disturbance; only the original system states are estimated. This is highly desirable since, in practice, one usually is only interested in estimating the original system states. We shall first consider the problem of estimating the system state vector, $\underline{x}_i$, where the noise term $\underline{w}_i$ is a Gauss-Markov sequence.

Again we consider the discrete system

$$\underline{x}_{i+1} = A_i\underline{x}_i + B_i\underline{u}_i + \underline{w}_i \tag{4.1}$$

$$\underline{y}_i = H_i\underline{x}_i + \underline{v}_i \tag{4.2}$$

The measurement noise, $\underline{v}_i$, is taken to be a Gaussian white sequence with covariance

$$E\{\underline{v}_i\underline{v}_j'\} = R_i\delta_{ij}$$

However, in the present case we model the plant disturbance, $\underline{w}_i$, as the output of a linear discrete system driven by a zero-mean Gaussian white sequence. The plant disturbance, $\underline{w}_i$, is therefore a zero-mean Gauss-Markov sequence generated as the output of the following system.

$$\underline{w}_{i+1} = \Gamma_i\underline{w}_i + \underline{\eta}_i \tag{4.3}$$

where $\underline{\eta}_i$ is a Gaussian white sequence. The covariance matrix of the noise

vector, $\underline{w}_{i+1}$, denoted as Q_{i+1}, is propagated sequentially according to the relation

$$Q_{i+1} = \Gamma_i Q_i \Gamma_i' + \overline{\underline{\eta}_i \underline{\eta}_i'} \tag{4.4}$$

As was done in the previous chapter, we will design a minimal order observer of the form

$$\underline{z}_{i+1} = F_i \underline{z}_i + G_i \underline{u}_i + D_i \underline{y}_i \tag{4.5}$$

where $\underline{z}_i$ is an (n-m)-dimensional vector and

$$\underline{z}_i = T_i \underline{x}_i + \underline{\varepsilon}_i \tag{4.6}$$

As before, the observer error evolves according to the recursive equation

$$\underline{\varepsilon}_{i+1} = F_i \underline{\varepsilon}_i + D_i \underline{v}_i - T_{i+1} \underline{w}_i \tag{4.7}$$

In this case, from the basic observer error equation (4.7), we obtain the observer error covariance

$$\begin{aligned} \overline{\underline{\varepsilon}_{i+1} \underline{\varepsilon}_{i+1}'} = {} & T_{i+1} (A_i P_i \overline{\underline{\varepsilon}_i \underline{\varepsilon}_i'} P_i' A_i' + A_i V_i R_i V_i' A_i' + Q_i) T_{i+1}' \\ & - T_{i+1} (A_i P_i \overline{\underline{\varepsilon}_i \underline{w}_i'} + \overline{\underline{w}_i \underline{\varepsilon}_i'} P_i' A_i') T_{i+1}' \end{aligned} \tag{4.8}$$

In obtaining (4.8) we have used the results that $F_i = T_{i+1} A_i P_i$ and $D_i = T_{i+1} A_i V_i$. At this point it is noted that the error covariance (4.8) is similar in form to the corresponding expression obtained for the white noise problem considered in the previous chapter [see equation (3.33)]. This suggests the possibility of applying the same observer design technique developed in the previous chapter to the present problem with sequentially correlated plant noise. However, when the plant noise is sequentially correlated, the observer error

covariance (4.8) contains extra terms due to the fact that the observer error at time "i" is correlated with the plant noise at time "i". Before proceeding with the observer design we shall digress momentarily to evaluate the cross correlation $\overline{\underline{\epsilon}_i \underline{w}_i'}$ needed in the solution of the observer error covariance (4.8).

From (4.7) we have

$$\underline{\epsilon}_i = F_{i,1}\,\underline{\epsilon}_1 + \sum_{j=1}^{i-1} F_{i,j+1}\, D_j \underline{v}_j - \sum_{j=1}^{i-1} F_{i,j+1}\, T_{j+1} \underline{w}_j \qquad i=2,3,\ldots \tag{4.9}$$

where we shall use the notation

$$F_{i,j} \triangleq \prod_{k=j}^{i-1} F_k$$

and $F_{i,i} \triangleq I$ for all "i". Initializing the observer as described previously in Chapter 3, the initial observer error becomes

$$\underline{\epsilon}_1 = -T_1 A_0(\underline{x}_0 - \overline{\underline{x}}_0) - T_1 \underline{w}_0 \tag{4.10}$$

Next, using the relationships

$$\begin{aligned} &E\{(\underline{x}_0 - \overline{\underline{x}}_0)\,\underline{w}_i{}'\} = 0 && \text{for all "i"} \\ &E\{\underline{v}_j \underline{w}_i{}'\} = 0 && \text{for all "i, j"} \end{aligned} \tag{4.11}$$

One obtains the result

$$\overline{\underline{\epsilon}_i \underline{w}_i{}'} = -\sum_{j=0}^{i-1} F_{i,j+1}\, T_{j+1}\, \overline{\underline{w}_j \underline{w}_i{}'} \qquad i=1,2,\ldots \tag{4.12}$$

From the solution to (4.3) which is

$$\underline{w}_i = \Gamma_{i,j}\,\underline{w}_j + \sum_{k=j}^{i-1} \Gamma_{i,k+1}\,\underline{\eta}_k \qquad i > j \tag{4.13}$$

one obtains the result

$$\overline{\underline{w}_j\,\underline{w}_i{}'} = \overline{\underline{w}_j\,\underline{w}_j{}'}\;\Gamma_{i,j}{}' = Q_j\,\Gamma_{i,j}{}' \tag{4.14}$$

where the covariance Q_j is obtained from (4.4). Substituting (4.14) into (4.12) gives the expression

$$\overline{\underline{\varepsilon}_i\,\underline{w}_i{}'} = -\sum_{j=0}^{i-1} F_{i,j+1}\,T_{j+1}\,Q_j\,\Gamma_{i,j}{}' \tag{4.15}$$

An extremely desirable property from the standpoint of filtering and processing of measurement data is the recursive nature of the filtering equations as, for example, in the Kalman filtering technique. Although (4.15) characterizes the cross correlation $\overline{\underline{\varepsilon}_i\underline{w}_i{}'}$, it is not in the desired recursive format. To obtain a recursive equation for $\overline{\underline{\varepsilon}_i\underline{w}_i{}'}$, consider expanding (4.15) as follows

$$\overline{\underline{\varepsilon}_{i+1}\,\underline{w}_{i+1}{}'} = -\sum_{j=0}^{i-1} F_{i+1,j+1}\,T_{j+1}Q_j\,\Gamma_{i+1,j} - T_{i+1}Q_i\,\Gamma_i{}' \tag{4.16}$$

Using the properties of the transition matrix, $F_{i,j}$, and the fact that $F_i = T_{i+1}A_iP_i$ it may be shown that (4.16) is of the form

$$\overline{\underline{\varepsilon}_{i+1}\,\underline{w}_{i+1}{}'} = T_{i+1}(A_iP_i\,\overline{\underline{\varepsilon}_i\underline{w}_i{}'} - Q_i)\,\Gamma_i{}'$$

where (4.17)

$$\overline{\underline{\varepsilon}_1\underline{w}_1{}'} = -T_1Q_o\Gamma_o{}'$$

Returning to the problem of designing an optimal observer for the system (4.1), (4.2), we again assume without loss of generality that the measurements are of the form

$$\underline{y}_i = [I_m | 0]\underline{x}_i + \underline{v}_i \tag{4.18}$$

Thus, the same observer structure used previously in Chapter 3 will be employed here. The observer output is therefore taken to be following

$$\underline{z}_i = [K_i | I_{n-m}]\underline{x}_i + \underline{\varepsilon}_i \tag{4.19}$$

where again we seek the optimal gain matrix, K_i^*, to minimize the overall mean square estimation error. Following closely the approach of Chapter 3, we begin by partitioning the observer error covariance (4.8) as follows.

$$\overline{\underline{\varepsilon}_{i+1}\underline{\varepsilon}_{i+1}'} = T_{i+1} \left[\begin{array}{c|c} \Omega_{11}^{\ i} & \Omega_{12}^{\ i} \\ \hline \Omega_{21}^{\ i} & \Omega_{22}^{\ i} \end{array} \right] T_{i+1}' \tag{4.20}$$

where the partitions of the matrix Ω_i are conformable with the partitioned matrix T_{i+1}. In the present case, the matrix Ω_i is defined by (4.21) below.

$$\Omega_i \triangleq A_i P_i \overline{\underline{\varepsilon}_i \underline{\varepsilon}_i'} P_i' A_i' + A_i V_i R_i V_i' A_i' + Q_i$$

$$- A_i P_i \overline{\underline{\varepsilon}_i \underline{w}_i'} - \overline{\underline{w}_i \underline{\varepsilon}_i'} P_i' A_i' \tag{4.21}$$

The next step in the observer design is to obtain the overall estimation error. From this point on the results are essentially identical in form to the white noise case considered in Chapter 3. Omitting the unnecessary details, we obtain the result

$$\operatorname{trace} \overline{\underline{e}_{i+1}\underline{e}_{i+1}'} = \operatorname{trace} R_{i+1}$$

$$+ \operatorname{trace} \{K_{i+1}(\Omega_{11}^{i}+R_{i+1})K_{i+1}' + K_{i+1}\Omega_{12}^{i} + \Omega_{21}^{i}K_{i+1}' + \Omega_{22}^{i}\} \tag{4.22}$$

Comparison of (4.22) with (3.47) of the previous chapter leads to the obvious conclusion that the optimal gain matrix is identical in form to that obtained for the white noise case. Hence, the optimal gain matrix is given by the expression

$$K_{i+1}^{*} = -\Omega_{21}^{i}(\Omega_{11}^{i} + R_{i+1})^{-1} \tag{4.23}$$

where in the case of a Gauss-Markov plant noise the computation of the matrix Ω_i is modified to account for the cross correlation between the observer error $\underline{\epsilon}_i$, and the plant noise, $\underline{w}_i$.

4.3 OBSERVER DESIGN FOR SYSTEMS WITH GAUSS-MARKOV MEASUREMENT NOISE

Next we shall consider the problem of sequential estimation of the state vector $\underline{x}_i$ of the plant (4.1), (4.2) using a minimal-order observer where the measurements are corrupted by a colored noise of the Gauss-Markov type. The plant noise, $\underline{w}_i$, is taken to be a Gaussian white sequence with covariance

$$E\{\underline{w}_i\underline{w}_j'\} = Q_i\delta_{ij} \tag{4.24}$$

However, here we model the measurement noise, $\underline{v}_i$, as a Gauss-Markov sequence generated as the output of the discrete system

$$\underline{v}_{i+1} = \theta_i\underline{v}_i + \underline{\xi}_i \tag{4.25}$$

$\underline{\xi}_i$ is a zero-mean Gaussian white sequence. The covariance matrix of the measurement noise, $\underline{v}_{i+1}$, denoted as R_{i+1}, evolves with time according to the relation

$$R_{i+1} = \theta_i R_i \theta_i{}' + \overline{\underline{\xi}_i \underline{\xi}_i{}'} \tag{4.26}$$

We shall next optimize our canonical observer design based upon the system model described above. From the basic observer error equation (4.7) and the fact that $F_i = T_{i+1}A_iP_i$ and $D_i = T_{i+1}A_iV_i$ it is easily shown that the observer error covariance is of the form

$$\overline{\underline{\epsilon}_{i+1}\underline{\epsilon}_{i+1}{}'} = T_{i+1}(A_iP_i\overline{\underline{\epsilon}_i\underline{\epsilon}_i{}'}P_i{}'A_i{}' + A_iV_iR_iV_i{}'A_i{}' + Q_i)\,T_{i+1}{}'$$

$$+ T_{i+1}(A_iP_i\,\overline{\underline{\epsilon}_i\underline{v}_i{}'}\,V_i{}'A_i{}' + A_iV_i\,\overline{\underline{v}_i\underline{\epsilon}_i{}'}P_i{}'A_i{}')\,T_{i+1}{}' \tag{4.27}$$

Noting that the observer error covariance (4.27) is essentially in the same form as (4.8), it is clear that the canonical observer structure used previously may be again utilized for the problem of colored measurement noise. Before proceeding with the observer design it will be necessary to obtain a recursive solution to the cross-covariance $\overline{\underline{\epsilon}_i\underline{v}_i{}'}$ needed in the evaluation of the observer error covariance (4.27). From the basic observer error equation (4.7) and the properties of the noises $\underline{w}_i$ and $\underline{v}_i$ [namely (4.24) and (4.25)] one obtains the result

$$\overline{\underline{\epsilon}_i\underline{v}_i{}'} = \sum_{j=1}^{i-1} F_{i,\,j+1}\,D_j\,\overline{\underline{v}_j\underline{v}_i{}'} \qquad i=2,3,\ldots \tag{4.28}$$

Since the observer is initialized as in (4.10) we have that

$$\overline{\underline{\epsilon}_1\underline{v}_1{}'} = 0 \tag{4.29}$$

From the solution of (4.25) which is

$$\underline{v}_i = \theta_{i,j}\underline{v}_j + \sum_{k=j}^{i-1} \theta_{i,k+1}\,\underline{\xi}_k \qquad i > j \tag{4.30}$$

where $\theta_{i,j} \triangleq \prod_{k=j}^{i-1} \theta_k$ and $\theta_{i,i} \triangleq I$ for all "i" we obtain the result

$$\overline{\underline{v}_j\underline{v}_i}' = \overline{\underline{v}_j\underline{v}_j}'\,\theta_{i,j}' = R_j\theta_{i,j}' \tag{4.31}$$

and the covariance R_j is obtained from (4.26). Substituting (4.31) into (4.28) gives the result

$$\overline{\underline{\epsilon}_i\underline{v}_i}' = \sum_{j=1}^{i-1} F_{i+1,j+1}\,D_jR_j\theta_{i+1,j}' + D_iR_i\theta_i' \tag{4.33}$$

Next using the properties of the transition matrix $F_{i,j}$ and the relationships $F_i = T_{i+1}A_iP_i$ and $D_i = T_{i+1}A_iV_i$ it may be shown that (4.33) is equivalent to the recursive expression

$$\overline{\underline{\epsilon}_{i+1}\underline{v}_{i+1}}' = T_{i+1}(A_iP_i\,\overline{\underline{\epsilon}_i\underline{v}_i}' + A_iV_iR_i)\,\theta_i'$$

where (4.34)

$$\overline{\underline{\epsilon}_1\underline{v}_1}' = 0$$

Without loss of generality we shall again assume the measurements to be of the form (4.18) and take the observer transformation to be of the form (4.19). As before, the observer design is optimized by obtaining the free gain matrix K_{i+1} which provides minimum overall mean square estimation error. The matrix Ω_i, defined below, is partitioned as described previously [see (4.20)].

$$\Omega_i = A_i P_i \overline{\underline{\varepsilon}_i \underline{\varepsilon}_i'} P_i' A_i' + A_i V_i R_i V_i' A_i' + Q_i$$

$$+ A_i P_i \overline{\underline{\varepsilon}_i \underline{v}_i'} V_i' A_i' + A_i V_i \overline{\underline{v}_i \underline{\varepsilon}_i'} P_i' A_i' \tag{4.35}$$

where $\overline{\underline{\varepsilon}_{i+1} \underline{\varepsilon}_{i+1}'} = T_{i+1} \Omega_i T_{i+1}'$.

Using (3.38), (3.44) the total estimation error covariance is found to be

$$\overline{\underline{e}_{i+1} \underline{e}_{i+1}'} = \left[\begin{array}{c|c} R_{i+1} & \overline{\underline{v}_{i+1} \underline{\varepsilon}_{i+1}'} - R_{i+1} K_{i+1}' \\ \hline \overline{\underline{\varepsilon}_{i+1} \underline{v}_{i+1}'} - K_{i+1} R_{i+1} & \begin{array}{c} \overline{\underline{\varepsilon}_{i+1} \underline{\varepsilon}_{i+1}'} \\ - \overline{\underline{\varepsilon}_{i+1} \underline{v}_{i+1}'} K_{i+1}' \\ - K_{i+1} \overline{\underline{v}_{i+1} \underline{\varepsilon}_{i+1}'} \\ + K_{i+1} R_{i+1} K_{i+1}' \end{array} \end{array}\right] \tag{4.36}$$

Before proceeding with the minimization of the mean square estimation error we shall rewrite the quantity $\overline{\underline{\varepsilon}_{i+1} \underline{v}_{i+1}'}$ in a more useful form. We partition (4.34) into the following form

$$\overline{\underline{\varepsilon}_{i+1} \underline{v}_{i+1}'} = T_{i+1} \left[\begin{array}{c} \pi_{11}^{\,i} \\ \hline \pi_{22}^{\,i} \end{array}\right] \tag{4.37}$$

where $\pi_{11}^{\,i}$ is the upper m x m dimensional partition and $\pi_{22}^{\,i}$ is the lower (n-m)xm dimensional partition. With this definition it can be shown that the mean square estimation error is given by

$$\text{trace}\ \overline{\underline{e}_{i+1}\underline{e}_{i+1}'} = \text{trace}\ R_{i+1}$$

$$+\ \text{trace}\ \left\{K_{i+1}\left(\Omega_{11}{}^{i} - \pi_{11}{}^{i} - \pi_{11}{}^{i\prime} + R_{i+1}\right)K_{i+1}\right. \tag{4.38}$$

$$\left. + K_{i+1}\left(\Omega_{12}{}^{i} - \pi_{22}{}^{i\prime}\right) + \left(\Omega_{21}{}^{i} - \pi_{22}{}^{i}\right)K_{i+1}{}' + \Omega_{22}{}^{i}\right\}$$

Setting the gradient of (4.38) with respect to the gain matrix K_{i+1} equal to zero gives the result

$$K_{i+1}\left(\Omega_{11}{}^{i} - \pi_{11}{}^{i} - \pi_{11}{}^{i\prime} + R_{i+1}\right) + \left(\Omega_{21}{}^{i} - \pi_{22}{}^{i}\right) = 0 \tag{4.39}$$

The minimizing solution is given by the following expression [1, 12, 24]

$$K_{i+1}{}^{*} = -\left(\Omega_{21}{}^{i} - \pi_{22}{}^{i}\right)\left(\Omega_{11}{}^{i} - \pi_{11}{}^{i} - \pi_{11}{}^{i\prime} + R_{i+1}\right)^{+} \tag{4.40}$$

where $(\ \ )^{+}$ is the Moore-Penrose pseudoinverse.

5. EXAMPLES ILLUSTRATING THE THEORY

5.1 INTRODUCTION

In this chapter we shall illustrate the application and utility of the observer design techniques developed in the preceeding chapters 3 and 4 of the dissertation. Toward this end we shall consider an important practical problem, namely the design of a radar tracking system (sometimes referred to as a track-while-scan radar system) based upon the previously developed theory of optimal minimal-order observers. In particular we treat two special cases and these are presented in the following sections of this chapter as examples 1 and 2. The purpose of these examples is to demonstrate in a clear and straightforward manner the usefulness of optimal minimal-order observer theory to an actual and realistic design problem. In the interest of simplicity we have selected target models for our examples which are sufficiently simple so that the resulting observer design equations are not too unwieldy and cumbersome. However, the target models will be sophisticated enough so that the results of this design study are realistic and provide useful design information in a real tracking situation.

In the first example we consider tracking targets having white noise acceleration inputs, that is, the target maneuver is a white noise sequence. The maneuver, therefore, at one sampling period is completely uncorrelated with the maneuver at a different sampling period. This situation prevails when the target exhibits constant velocity except for random disturbances. Also, the measurement errors are assumed to be independent from measurement to measurement. Typically, ballistic missiles, orbital and suborbital

targets are modeled in this way. Example 1 is intended to demonstrate the basic optimal minimal-order observer design for systems having white noise distrubances as treated in Chapter 3.

In the sequel we shall, of course, compare the resulting performance of the best minimal-order observer tracking system with the performance obtained from the corresponding theoretically optimal Kalman filter tracking system. Also, in our comparative study we shall investigate the constant eigenvalue observer designs of Dellon [10] and Williams [32] and we shall compare the performance of these designs with the best minimal-order observer design.

In the second example we treat a slightly more sophisticated (and perhaps more realistic) target model, namely the case where target acceleration is characterized as a time-wise correlated noise sequence. Physically speaking, this is interpreted as the situation where if the target being tracked is accelerating (maneuvering) at time instant "i" then it is also likely to be accelerating (maneuvering) at the next observation time instant "i+1." Typically, manned maneuvering targets such as aircraft, ships and submarines are generally modeled in this way [27]. The maneuver properties of a particular target are characterized, therefore, by two parameters, and these parameters are the target maneuver variance and correlation time or time constant. In the second example we shall treat the maneuver variance as constant and shall vary the maneuver correlation time in parametric fashion. Hence the resulting tracking accuracy of the best minimal-order observer tracker and the Kalman tracker is, for the most part, presented graphically. In this way a large class of manned maneuvering targets is considered and the performance of the tracking system

for any single particular target is obtained from the graphs by specifying its particular maneuver properties. The purpose of example 2 is to demonstrate the application of our optimal minimal-order observer design technique for the case of systems with time-wise correlated noise inputs as discussed in Chapter 4 of the dissertation.

5.2 EXAMPLE 1

To illustrate the application of minimal order-observer theory in a practical design situation we shall consider the following standard radar tracking problem. For purposes of simplicity we shall treat only the special case of a single spatial dimension. In particular, the target motion is confined to motion along the x-axis of the usual cartesian coordinate axes and the radar is assumed to provide range measurements along this same x-axis. Mathematically the target equations of motion for this simplified one-dimensional radar tracking situation are given in state variable representation by the following [28]:

$$\underbrace{\begin{bmatrix} x_{i+1} \\ \dot{x}_{i+1} \\ \ddot{x}_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ \dot{x}_i \\ \ddot{x}_i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_i}_{\underline{x}_{i+1} = A_i \underline{x}_i + \underline{w}_i} \tag{5.1}$$

$$
\underbrace{y_i = [1 \quad 0 \quad 0] \begin{bmatrix} x_i \\ \dot{x}_i \\ \ddot{x}_i \end{bmatrix} + v_i}_{\underline{y}_i = H_i \underline{x}_i + \underline{v}_i} \tag{5.2}
$$

As indicated in (5.2), the position of the target along the x-axis is measured by the ground radar. The measurements contain observation noise which is represented by an additive zero-mean Gaussian white sequence, v_i, having variance σ_v^2 (measurement noise variance). Practically speaking, the radar measurement error would be range dependent. However, in this simplified example we shall take the variance, σ_v^2, to be constant. In (5.1), the input w_i represents the change in target acceleration from time "i" to time "i+1" and for purposes of this example w_i is assumed to be a zero-mean Gaussian white sequence with variance σ_m^2 (maneuver variance). The data rate, T, is assumed to be constant so that target position is observed every T seconds.

One additional comment concerning the observability properties of this system is appropriate at this time. It is clear that the system (5.1), (5.2) (defined by the pair of matrices $(A, \underline{h})$) is observable in the sense of Kalman [16]. Checking the rank of the observability matrix we obtain the result

$$
\mathrm{Det} \begin{bmatrix} \underline{h} \\ \underline{h}A \\ \underline{h}A^2 \end{bmatrix} = \mathrm{Det} \begin{bmatrix} 1 & 0 & 0 \\ 1 & T & T^2/2 \\ 1 & 2T & 2T^2 \end{bmatrix} = T^3 \tag{5.3}
$$

Hence the system defined by $(A, \underline{h})$ is observable in the usual sense for all data rates $T > 0$.

Kalman's filter for the system (5.1), (5.2) is a 3-state filter defined by the following equations:

$$\hat{\underline{x}}_{i+1/i+1} = \hat{\underline{x}}_{i+1/i} + K_{i+1} (\underline{y}_{i+1} - H_{i+1} \hat{\underline{x}}_{i+1/i}) \tag{5.4}$$

where Kalman's gain matrix is

$$K_{i+1} = P_{i+1/i} H_{i+1}' (H_{i+1} P_{i+1/i} H_{i+1}' + R_{i+1})^{-1} \tag{5.5}$$

and

$$\hat{\underline{x}}_{i+1/i} = A_i \hat{\underline{x}}_{i/i} \tag{5.6}$$

The n-vector $\hat{\underline{x}}_{i+1/i+1}$ is the minimum mean square estimate of $\underline{x}_{i+1}$ given measurements up-to and including time "i+1" (i.e., the filtered estimate) and $\hat{\underline{x}}_{i+1/i}$ is the minimum mean square estimate of $\underline{x}_{i+1}$ given measurements up-to and including time "i" (i.e., the one-step-ahead prediction). The $n \times n$ matrices $P_{i+1/i+1}$ and $P_{i+1/i}$ are the covariance matrices of the filtered and one-step-ahead prediction errors, respectively. These matrices satisfy the following recursive equations.

$$\begin{aligned} P_{i+1/i} &= A_i P_{i/i} A_i + Q_i \\ P_{i+1/i+1} &= (I_n - K_{i+1} H_{i+1}) P_{i+1/i} \end{aligned} \tag{5.7}$$

Design of Kalman's optimal linear filter is essentially complete at this point. The structure of the filter is given in equations (5.4) through (5.6) and initialization of the filter is performed in accordance with (5.7).

Finally, following the approach taken by Singer and Monzingo [28], we shall initialize the Kalman filter equations by taking as the initial state estimate

$$
\begin{aligned}
\hat{x}_{o/o} &= y_o \\
\hat{\dot{x}}_{o/o} &= \frac{1}{T}\left(\frac{3}{2}\, y_o - 2y_{-1} + \frac{1}{2}\, y_{-2}\right) \qquad (5.8) \\
\hat{\ddot{x}}_{o/o} &= \frac{1}{T^2}\,(y_o - 2y_{-1} + y_{-2})
\end{aligned}
$$

where y_{-2}, y_{-1} and y_o are, respectively, the first, second and third radar measurements received. The corresponding covariance initialization equation for (5.8) is given by the following:

$$
P_{o/o} = \left[\begin{array}{c|c|c}
\sigma_v^2 & \frac{3}{2}\,\frac{\sigma_v^2}{T} & \frac{\sigma_v^2}{T^2} \\
\hline
\frac{3}{2}\,\frac{\sigma_v^2}{T} & \frac{13}{2}\,\frac{\sigma_v^2}{T^2} + \frac{T^2\sigma_m^2}{16} & \frac{6\sigma_v^2}{T^3} + \frac{\sigma_m^2 T}{8} \\
\hline
\frac{\sigma_v^2}{T^2} & \frac{6\sigma_v^2}{T^3} + \frac{\sigma_m^2 T}{8} & \frac{6\sigma_v^2}{T^4} + \frac{5\sigma_m^2}{4}
\end{array}\right] \qquad (5.9)
$$

Since Kalman's linear filter provides the best attainable performance in terms of minimizing the mean-square estimation error, it will provide us with a useful upper bound to tracking filter performance. Hence, our purpose in presenting the Kalman filter here is to provide a reference against which the performance of our minimal-order observer may be compared. We shall next present the design equations for the minimal-order observer.

Design of the optimal minimal-order observer for the system described by $(A, \underline{h})$ is relatively straightforward and involves evaluating the design equations derived in Chapter 3. We note at this point that the state equations for the system considered in this example are already in the desired observer canonical form (that is, transformation of the state equations to a new coordinate system is unnecessary, and therefore the basic design equations of Chapter 3 apply without modification. For convenience we tabulate the appropriate design equations below.

$$T_i = [K_i | I_{n-m}] \tag{5.10}$$

$$P_i = \left[\frac{0}{I_{n-m}}\right] \quad , \quad V_i = \left[\frac{I_m}{-K_i}\right] \tag{5.11}$$

$$K_{i+1} = -\Omega_{21}^{i}(\Omega_{11}^{i} + R_{i+1})^{-1} \tag{5.12}$$

$$F_i = A_{22}^{i} + K_{i+1} A_{12}^{i} \tag{5.13}$$

$$D_i = A_{21}^{i} - A_{22}^{i} K_i + K_{i+1}(A_{11}^{i} - A_{12}^{i} K_i) \tag{5.14}$$

We shall present next the solution to the minimal-order observer equations given above. Since the system defined by $(A, \underline{h})$ has n=3 state variables and m=1 output measurement, the dimension of the minimal-order observer is n-m=2 and therefore the observer transformation, T_i, satisfies the relationship

$$\underline{z}_i = T_i \underline{x}_i + \underline{\varepsilon}_i$$

where

$$T_i = \left[\begin{array}{c|cc} k_i^{(1)} & 1 & 0 \\ k_i^{(2)} & 0 & 1 \end{array}\right] \qquad (5.15)$$

$$\underbrace{\quad}_{K_i} \quad \underbrace{\qquad}_{I_2}$$

Hence, T_i is a 2 x 3 rectangular matrix containing the arbitrary gain elements $k_i^{(1)}$ and $k_i^{(2)}$. These arbitrary gain elements are adjusted in an adaptive manner to minimize the overall mean-square estimation error at each time instant "i." Computation of the corresponding P_i and V_i matrices results in the following:

$$P_i = \left[\begin{array}{cc} 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] , \quad V_i = \left[\begin{array}{c} 1 \\ \hline -k_i^{(1)} \\ -k_i^{(2)} \end{array}\right] \qquad (5.16)$$

The estimate of the state vector $\underline{x}_i$ is, of course, given by the following

$$\hat{\underline{x}}_i = P_i \, \underline{z}_i + V_i \, \underline{y}_i \qquad (5.17)$$

with P_i and V_i as defined in (5.16). Next, the observer transition matrix, F_i, is found to be for this example

$$F_i = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A_{22}^i} + \underbrace{\begin{bmatrix} k_{i+1}^{(1)} \\ k_{i+1}^{(2)} \end{bmatrix}}_{K_{i+1}} \underbrace{\begin{bmatrix} T & \frac{T^2}{2} \end{bmatrix}}_{A_{12}^i} \qquad (5.18)$$

$$D_i = \underbrace{\begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}}_{A_{21}^{\,i}} - \underbrace{\begin{bmatrix} 1 & T \\ \\ 0 & 1 \end{bmatrix}}_{A_{22}^{\,i}} \underbrace{\begin{bmatrix} k_i^{(1)} \\ \\ k_i^{(2)} \end{bmatrix}}_{K_i} + \underbrace{\begin{bmatrix} k_{i+1}^{(1)} \\ \\ k_{i+1}^{(2)} \end{bmatrix}}_{K_{i+1}} \left(\underbrace{[1]}_{A_{11}^{\,i}} - \underbrace{\begin{bmatrix} T & \frac{T^2}{2} \end{bmatrix}}_{A_{12}^{\,i}} \underbrace{\begin{bmatrix} k_i^{(1)} \\ \\ k_i^{(2)} \end{bmatrix}}_{K_i} \right) \tag{5.19}$$

Since the three defining matrices (T_i, F_i, D_i) have been specified uniquely in terms of the unknown adaptive gain elements $k_i^{(1)}$ and $k_i^{(2)}$ [see equations (5.15), (5.18) and (5.19)], design of the basic observer structure is essentially complete. It remains only to specify the computation of the optimal gain matrix K_i (that is, the optimal gain elements $k_i^{(1)}$ and $k_i^{(2)}$) and to describe the observer initialization technique.

Determination of the optimal observer gain matrix, K_{i+1}, is a recursive procedure which uses the covariance matrix $\Omega_i = A_iP_i\,\overline{\underline{\varepsilon}_i\underline{\varepsilon}_i{}'}\,P_i{}'A_i{}' + A_iV_iR_iV_i{}'A_i{}' + Q_i$. For the system defined by the state equations (5.1), (5.2) the matrix Ω_i is found to be

$$\Omega_i = \underbrace{\begin{bmatrix} T & T^2/2 \\ 1 & T \\ 0 & 1 \end{bmatrix}}_{A_iP_i} \overline{\underline{\varepsilon}_i\underline{\varepsilon}_i{}'} \underbrace{\begin{bmatrix} T & 1 & 0 \\ \\ T^2/2 & T & 1 \end{bmatrix}}_{P_i{}'A_i{}'} \tag{5.20}$$

$$+ \underbrace{\begin{bmatrix} 1-k_i^{(1)} - \frac{k_i^{(2)}T^2}{2} \\ -k_i^{(1)} - k_i^{(2)}T \\ -k_i^{(2)} \end{bmatrix}}_{A_iV_i} \underbrace{\sigma_v^{\,2}}_{R_i} \underbrace{\begin{bmatrix} 1-k_i^{(1)} - \frac{k_i^{(2)}T^2}{2}, & -k_i^{(1)} - k_i^{(2)}T, & -k_i^{(2)} \end{bmatrix}}_{V_i{}'A_i{}'}$$

$$+\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_m^2 \end{bmatrix}}_{Q_i} \qquad (5.20) \text{ Cont.}$$

The gain matrix K_{i+1} is obtained from the relation $K_{i+1} = -\Omega_{21}^{\,i}(\Omega_{11}^{\,i} + R_{i+1})^{-1}$ where $\Omega_{21}^{\,i}$ and $\Omega_{11}^{\,i}$ are the appropriate partitions of the covariance Ω_i and R_{i+1} is the measurement noise covariance at time "i+1." Let the observer error covariance be the following

$$\overline{\underline{\varepsilon}_i \underline{\varepsilon}_i'} \triangleq \begin{bmatrix} \varepsilon_{11}^{\,i} & \varepsilon_{12}^{\,i} \\ \varepsilon_{12}^{\,i} & \varepsilon_{22}^{\,i} \end{bmatrix} \qquad (5.21)$$

and partition the matrix Ω_i in (5.9) in the form

$$\Omega_i \triangleq \left[\begin{array}{c|cc} \omega_{11}^{\,i} & \omega_{12}^{\,i} & \omega_{13}^{\,i} \\ \hline \omega_{12}^{\,i} & \omega_{22}^{\,i} & \omega_{23}^{\,i} \\ \omega_{13}^{\,i} & \omega_{23}^{\,i} & \omega_{33}^{\,i} \end{array}\right] \qquad (5.22)$$

Finally, the optimal gain elements, $k_{i+1}^{(1)}$ and $k_{i+1}^{(2)}$, can be written in closed form as follows

$$k_{i+1}^{(1)} = -\frac{\omega_{12}^{\,i}}{\omega_{11}^{\,i} + \sigma_v^2} \qquad (5.23)$$

$$k_{i+1}^{(2)} = - \frac{\omega_{13}^{i}}{\omega_{11}^{i} + \sigma_v^2} \tag{5.24}$$

where

$$\omega_{11}^{i} = T^2\epsilon_{11}^{i} + T^3\epsilon_{12}^{i} + \frac{T^4\epsilon_{22}^{i}}{4}$$

$$+ \sigma_v^2\left(1 - Tk_i^{(1)} - \frac{T^2}{2}k_i^{(2)}\right)^2$$

$$\omega_{12}^{i} = T\epsilon_{11}^{i} + \frac{3T^2\epsilon_{12}^{i}}{2} + \frac{T^3\epsilon_{22}^{i}}{2}$$

$$- \sigma_v^2\left(1 - Tk_i^{(1)} - \frac{T^2k_i^{(2)}}{2}\right)\left(k_i^{(1)} + Tk_i^{(2)}\right) \tag{5.25}$$

$$\omega_{13}^{i} = T\epsilon_{12}^{i} + \frac{T^2\epsilon_{22}^{i}}{2}$$

$$- \sigma_v^2\left(1 - Tk_i^{(1)} - \frac{T^2k_i^{(2)}}{2}\right)\left(k_i^{(2)}\right)$$

Initialization of the observer requires the evaluation of the covariance matrix $\Omega_o \overset{\Delta}{=} A_oM_oA_o' + Q_o$ where for this example, since $M_o = P_{o/o}$, we obtain the result:

$$
\Omega_o = \left[\begin{array}{c|c|c}
19\sigma_v^2 + \frac{T^4\sigma_m^2}{2} & \frac{21\sigma_v^2}{T} + \frac{7T^3\sigma_m^2}{8} & \frac{10\sigma_v^2}{T^2} + \frac{3T^2\sigma_m^2}{4} \\
\hline
\frac{21\sigma_v^2}{T} + \frac{7T^3\sigma_m^2}{8} & \frac{49\sigma_v^2}{2T^2} + \frac{25T^2\sigma_m^2}{16} & \frac{12\sigma_v^2}{T^3} + \frac{11T\sigma_m^2}{8} \\
\hline
\frac{10\sigma_v^2}{T^2} + \frac{3T^2\sigma_m^2}{4} & \frac{12\sigma_v^2}{T^3} + \frac{11T\sigma_m^2}{8} & \frac{6\sigma_v^2}{T^4} + \frac{9\sigma_m^2}{4}
\end{array}\right] \tag{5.26}
$$

Finally, the optimal initializing gain elements, $k_1^{(1)}$ and $k_1^{(2)}$, are found to be:

$$
k_1^{(1)} = \frac{\frac{21\sigma_v^2}{T} + \frac{7T^3\sigma_m^2}{8}}{19\sigma_v^2 + \frac{\sigma_m^2 T^4}{2} + \sigma_v^2} \tag{5.27}
$$

$$
k_1^{(2)} = - \frac{\frac{10\sigma_v^2}{T^2} + \frac{3T^2\sigma_m^2}{4}}{19\sigma_v^2 + \frac{\sigma_m^2 T^4}{2} + \sigma_v^2}
$$

5.3 PERFORMANCE EVALUATION, EXAMPLE 1

We shall present next the results of a comparative study of several tracking system designs for tracking targets as modeled in example 1. Among those tracking filters evaluated are included the Kalman filter, the optimal minimal-order observer, the optimal steady-state minimal order observer, and the constant eigenvalue observer designs of Dellon [10] and Williams [32]. A comparison of the tracking accuracy for these several tracking systems

is presented graphically in figures 5.1 through 5.7. Before discussing these computer results, the following descriptive comments are necessary:

1. The optimal steady-state minimal-order observer is identical in structure to the optimal (time-varying) minimal-order observer design developed in this dissertation, with the exception that the observer gain matrix is constant and equal to the steady-state gain matrix, $\lim_{i \to \infty} K_i^*$, obtained from the minimal-order observer algorithms.

2. Dellon's constant eigenvalue observer design is also identical in structure to the optimal minimal-order observer.* However, in this design the observer gain matrix is chosen to yield a fixed time-invariant observer with two constant and equal eigenvalues. Hence, to design a Dellon-type observer for this example it is necessary to determine the observer gain matrix, K, such that the observer F matrix, where $F = A_{22} + KA_{12}$, has the characteristic equation $P(\lambda) = (\lambda - \lambda_o)^2$ and λ_o is the desired observer eigenvalue. This observer is therefore completely specified by its eigenvalue, λ_o. The solution is easily shown to be the following.

$$F = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A_{22}} + \underbrace{\begin{bmatrix} k^{(1)} \\ k^{(2)} \end{bmatrix}}_{K} \underbrace{[T \quad T^2/2]}_{A_{12}} \tag{5.28}$$

with

---

*Dellon's work is discussed in Section 2.1, Chapter 2.

$$k^{(1)} = \frac{\lambda_o^2 + 2\lambda_o - 3}{2T}$$

and

$$k^{(2)} = \frac{2\lambda_o - \lambda_o^2 - 1}{T^2}$$

3. Williams' constant eigenvalue observer design is identical in structure to a Kalman filter except instead of implementing Kalman's gain matrix the observer gain matrix is chosen to yield an observer with three constant and equal eigenvalues.* To design a Williams-type observer for this example it is necessary to determine the triple of matrices (T, F, D) satisfying the fundamental observer equation TA = FT + DHA such that the observer F matrix has the characteristic equation $P(\lambda) = (\lambda - \lambda_o)^3$ and λ_o is the desired observer eigenvalue. Hence, the Williams' observer is also completely specified by its eigenvalue, λ_o. For the system (A, H) of example 1 the solution is found to be the following.

$$F = \begin{bmatrix} 3\lambda_o & 1 & 0 \\ -3\lambda_o^2 & 0 & 1 \\ \lambda_o^3 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 3-3\lambda_o \\ 3\lambda_o^2-3 \\ 1-\lambda_o^3 \end{bmatrix} \tag{5.29}$$

*Williams' work is discussed in Section 2.2, Chapter 2.

$$T = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ -2 & -T & \frac{T^2}{2} \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{matrix}(5.29)\\ \text{Cont.}\end{matrix}$$

Having described each of the observer designs considered in this comparative study we are now ready to discuss the computer results presented graphically in Figures 5.1 through 5.7. In this study, the following typical radar and target model parameters were used:

1. Radar range measurement accuracy, $\sigma_v = 10$ (ft.)

2. Target maneuver variance, $\sigma_m^2 = 100$ (ft./sec.^{2})^2

3. Data Rate, T = 1 second

Presented in Figures 5.1 and 5.2 is the total mean-square estimation error versus the discrete time index "i" (that is, the trace $\{(\underline{x}_i - \hat{\underline{x}}_i) \cdot (\underline{x}_i - \hat{\underline{x}}_i)'\}$ versus time "i"). Figure 5.1 demonstrates the results Dellon's design for observing eigenvalues of $\lambda = .3, .4, .45$ and $.5$ and also demonstrates the results of the Kalman filter, the optimal observer* and the optimal steady-state observer. With reference to Figure 5.1, it is clear that the overall steady-state estimation error of the optimal observer is increased from that of the Kalman filter by approximately 5.9% whereas for the best possible equal eigenvalue design ($\lambda_b = .45$) the corresponding

*For the sake of brevity we shall refer to the "optimal minimal-order observer" as the "optimal observer."

degredation is on the order of 16.5%. Therefore it is concluded that the steady-state performance of the optimal observer is superior, by far, to the best equal eigenvalue observer design. Inspection of the transient behavior also shows this same general trend to be true, as seen in Figure 5.1. (Note also, in this example, that the optimal steady-state observer provides excellent tracking performance, not only in the steady-state but during the transient period as well).

Another interesting comment can be made concerning the results of Figure 5.1. In viewing the results of Figure 5.1 it is seen that the best steady state performance is achieved with $\lambda_0 = .45$, however during the transient period the design with $\lambda_0 = .4$ performs best indicating that to obtain acceptable tracking performance (during both the transient period and in the steady-state) based on selection of observer eigenvalues it is perhaps necessary to select the eigenvalue in an adaptive manner. This idea was first proposed by Bona [7] where it was suggested that the response time could be decreased by using one eigenvalue during the transient period and after a given time the eigenvalue could be increased to improve steady-state estimation accuracy.

Similar comments can be made about the performance of the Williams' 3-state observer design as seen from Figure 5.2. To achieve the best steady-state tracking performance in this case, one takes the observer eigenvalue to be $\lambda_0 = .35$. However, it is seen in Figure 5.2 that $\lambda_0 = .3$ provides much better tracking accuracy during the transient period. In regards to steady-state tracking performance it is seen that for the best eigenvalue ($\lambda_0 = .35$) the overall mean-square error is increased by approximately 10.7% from that of the Kalman filter.

Figures 5.3 through 5.7 provide a breakdown of the overall mean-square estimation error into target position, velocity and acceleration errors. Figure 5.3 shows the mean-square error in the position estimate versus discrete time "i" for each of the observer designs being evaluated. Since Williams' observer is a 3-state filter, it provides some improvement in the estimate of target position whereas the minimal-order observer designs (including the optimal observer, the optimal steady-state observer and Dellon's equal eigenvalue observer) do not improve the accuracy in target position. This is no great loss however, since even the Kalman filter only improves the accuracy in target position from its initial value of 10 feet r.m.s. to approximately 9 feet r.m.s. in the steady state. From the standpoint of good tracking system design this slight improvement in position accuracy is hardly worth the effort. Reduction in the size of the tracking filter from 3 states to 2 states will result in significantly reduced computer processing requirements while yielding only a slight loss in position accuracy.

Figures 5.4 and 5.5 present the corresponding mean-square error in the estimate of target velocity. From these curves one obtains the relative degredation in the velocity estimate (ft./sec.) from that of the Kalman filter to be, in the steady-state, 3.3% for the optimal observer, 6.2% for the Williams observer with $\lambda_0 = .35$ and 11.2% for the Dellon observer with $\lambda_0 = .45$. Similar comments can be made concerning the mean-square errors in the estimate of target acceleration shown in Figures 5.6 and 5.7.

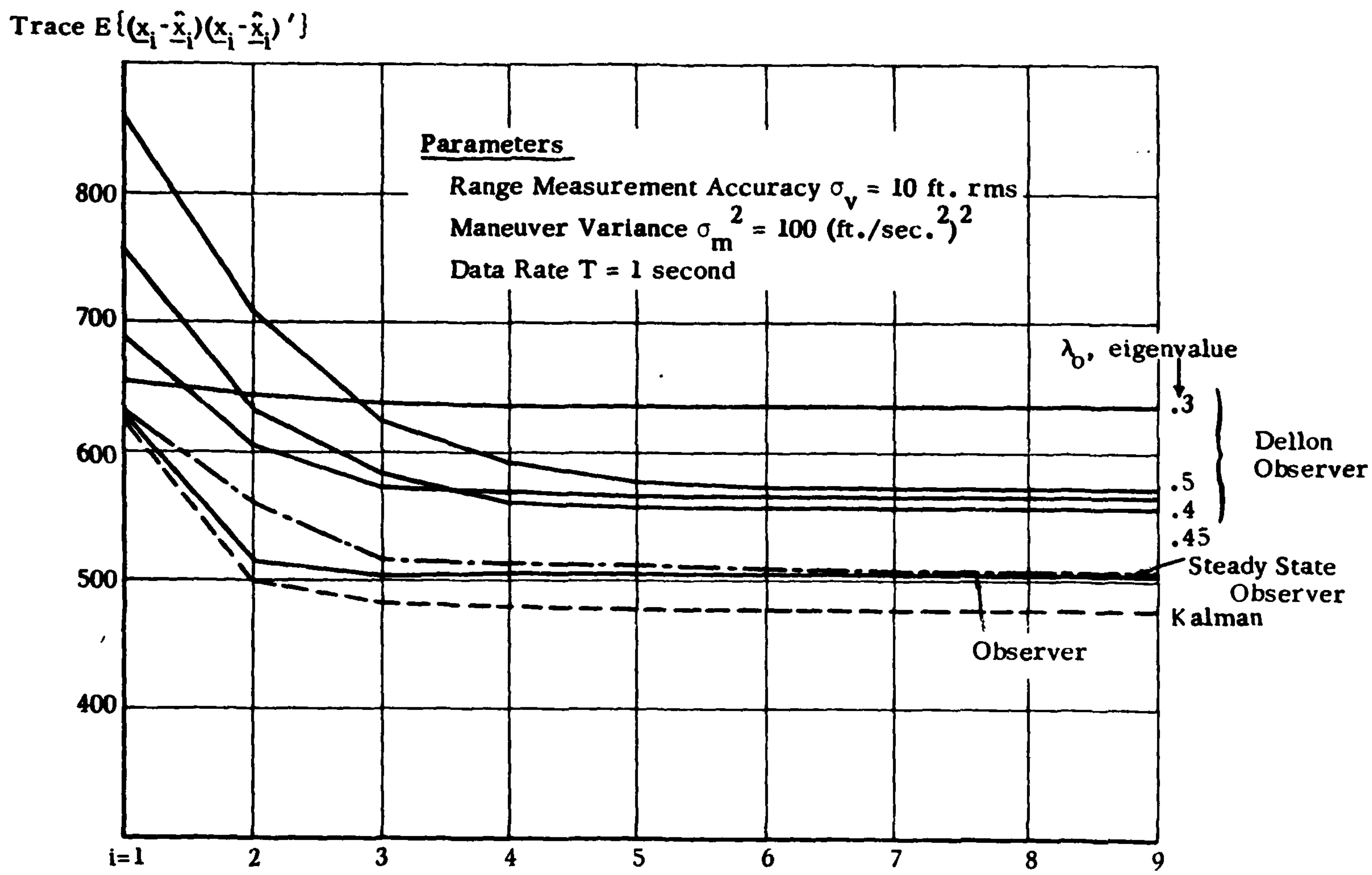


Figure 5.1 Total Mean-Square Estimation Error vs. Discrete Time "i."

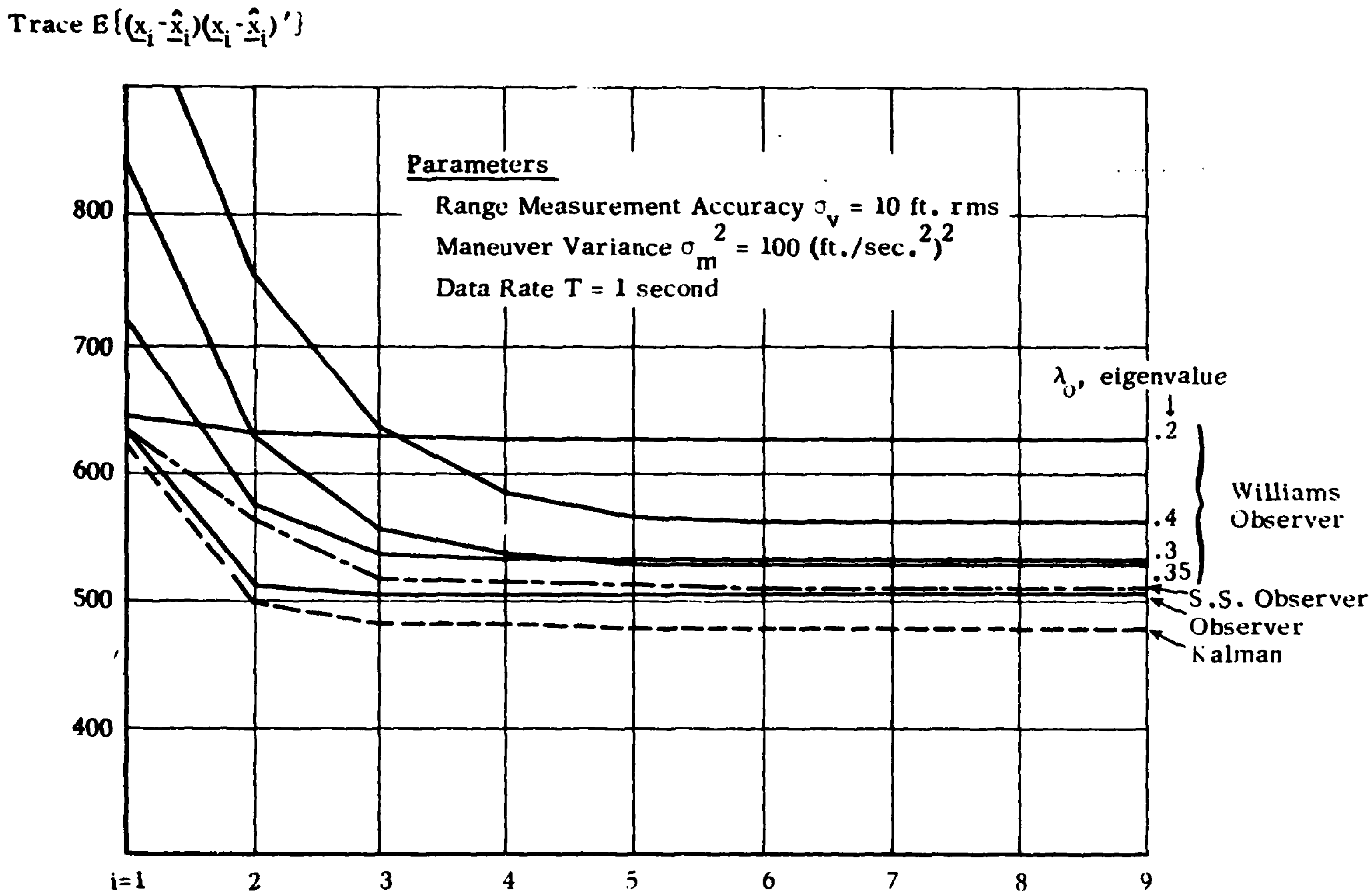


Figure 5.2 Total Mean-Square Estimation Error vs. Discrete Time "i."

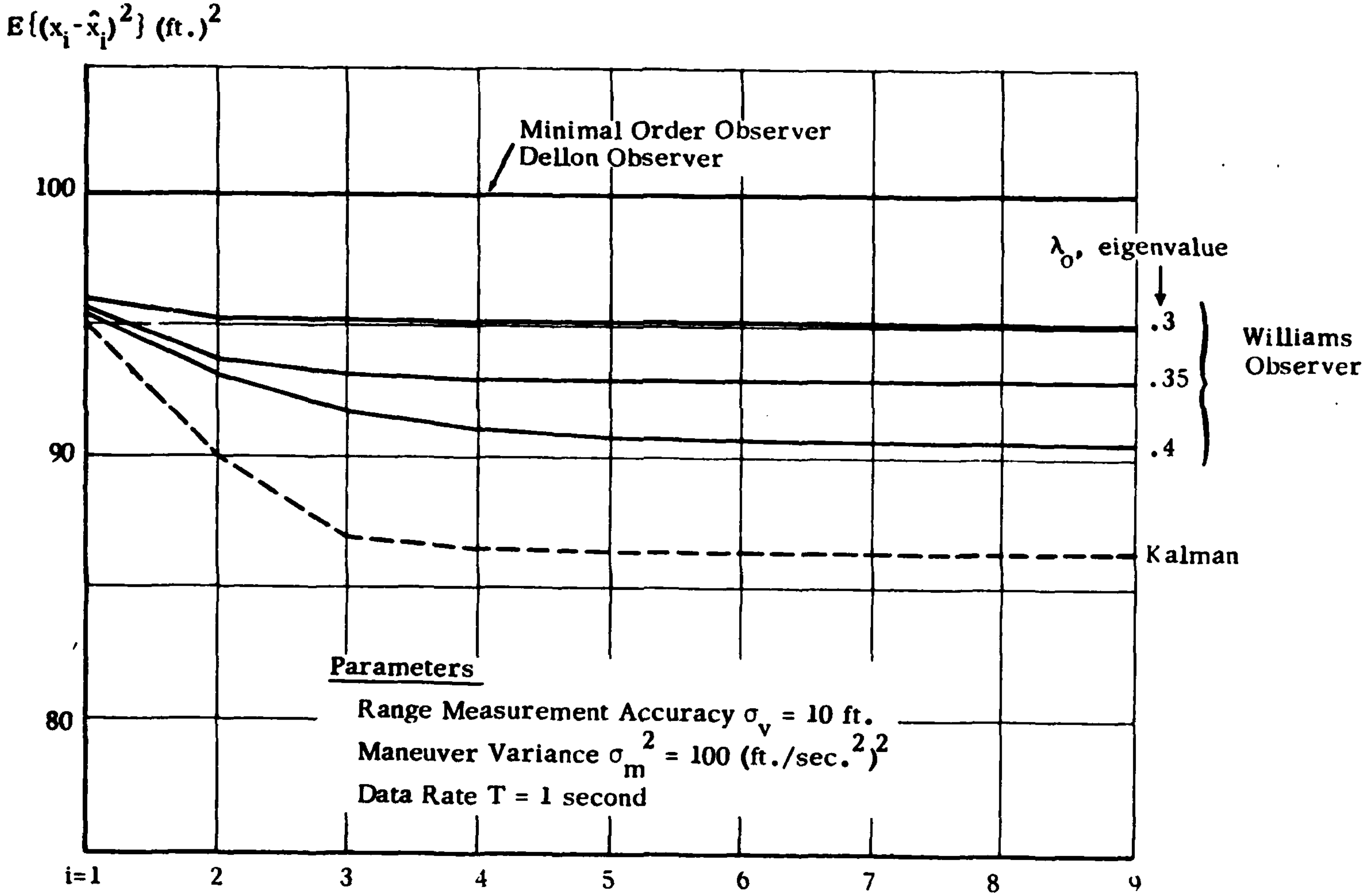


Figure 5.3 Mean-Square Error in Position vs. Discrete Time "i."

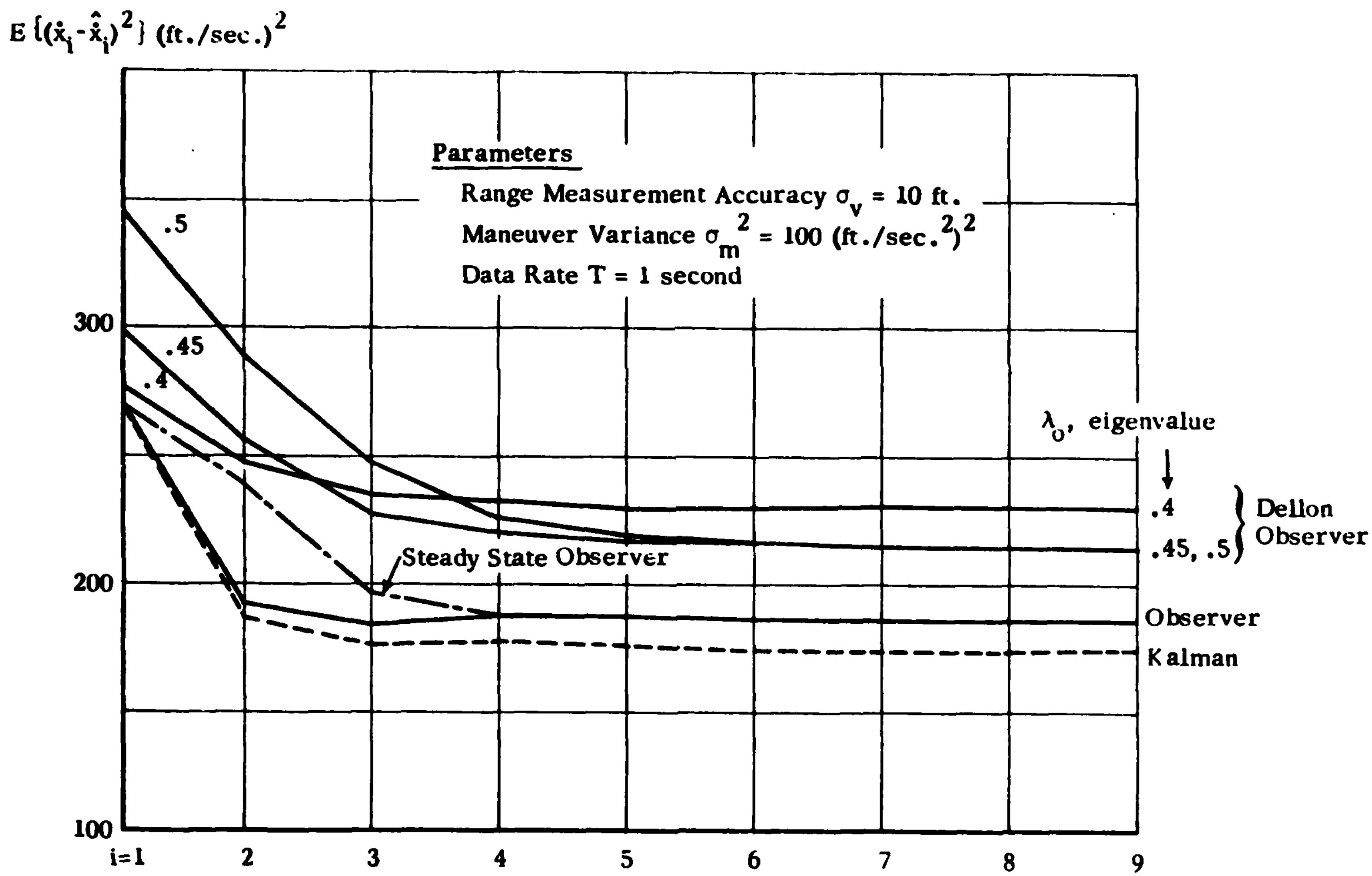


Figure 5.4 Mean-Square Error in Velocity vs. Discrete Time "i."

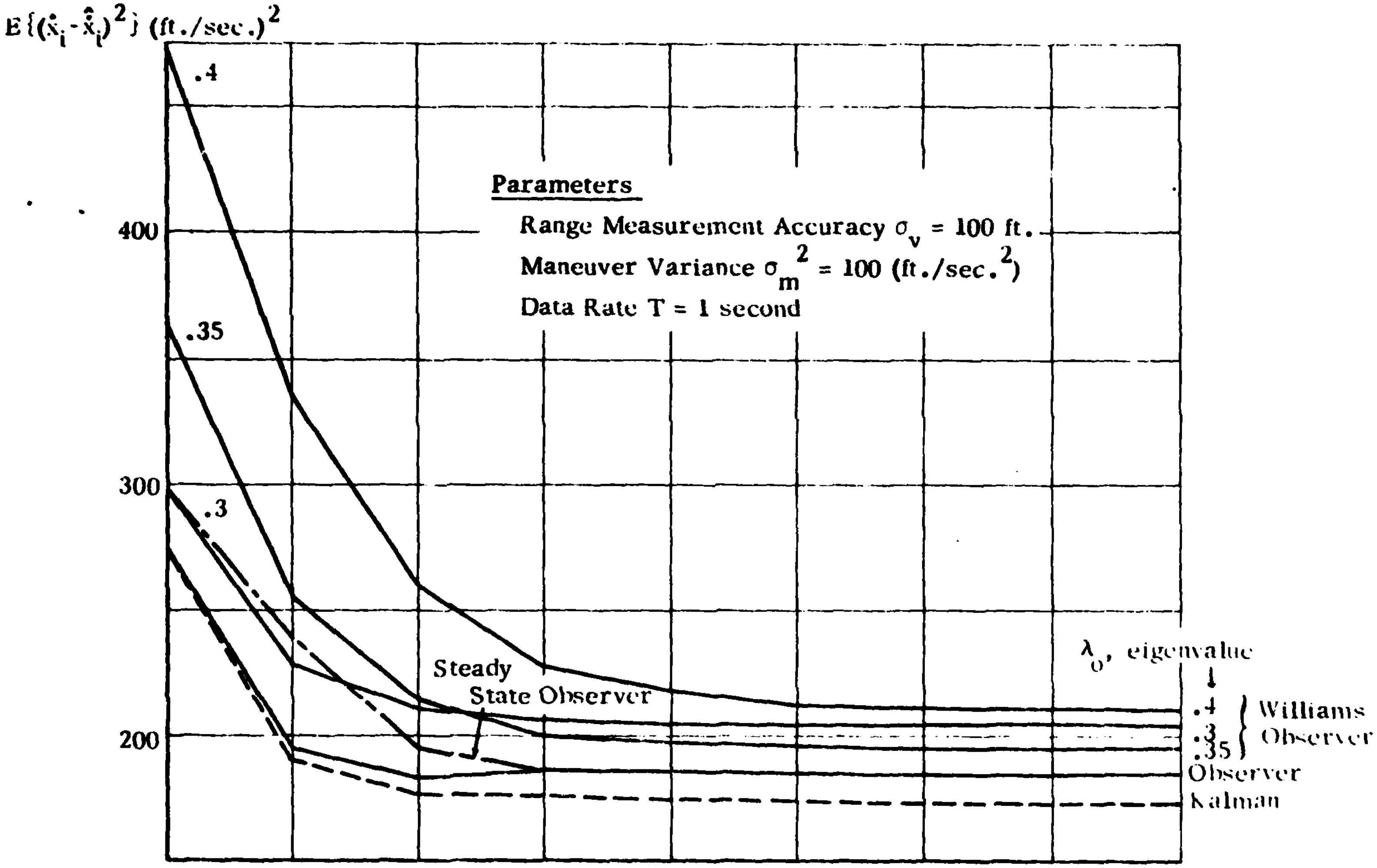


Figure 5.5 Mean-Square Error in Velocity vs. Discrete Time "i."

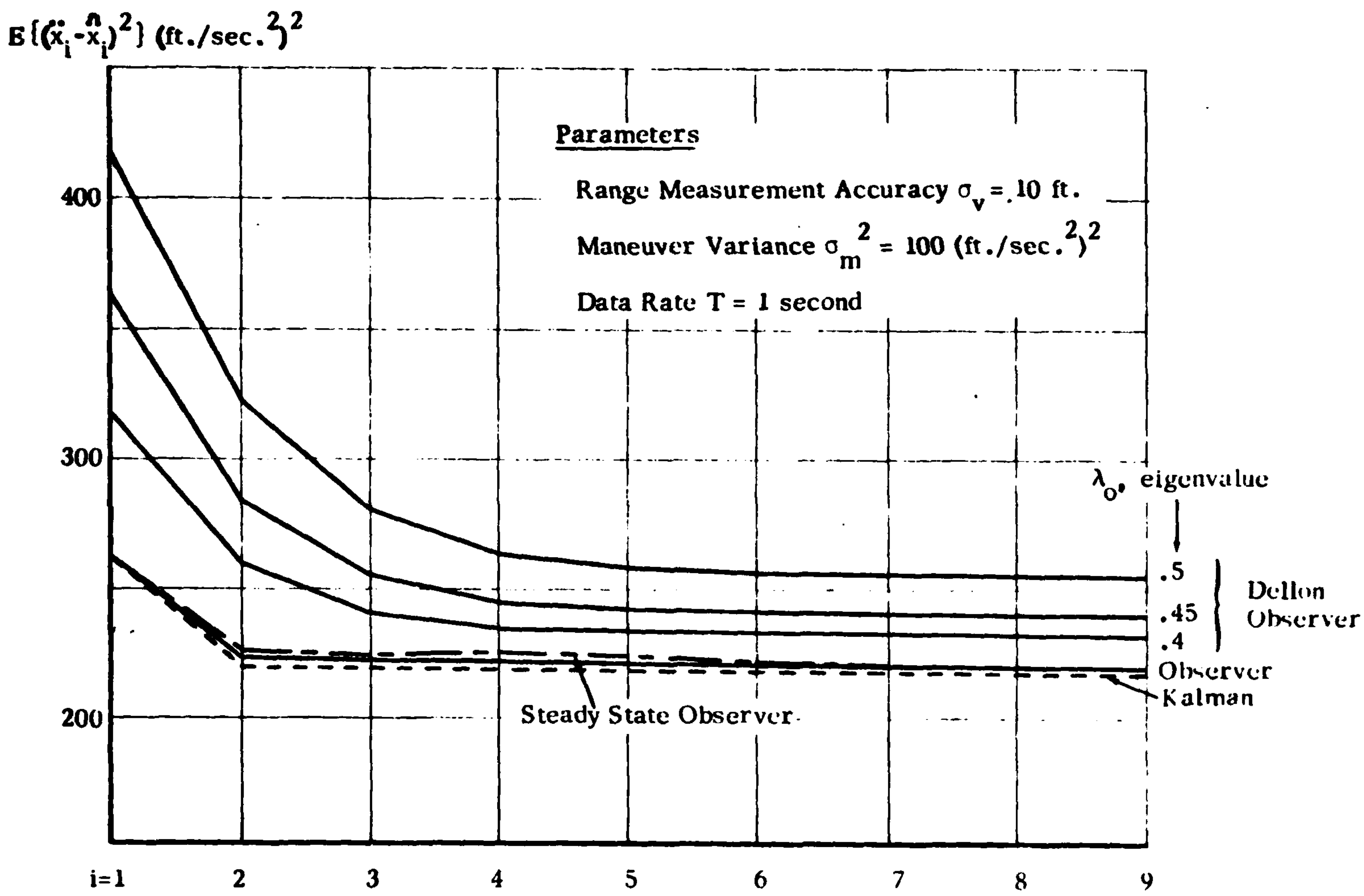


Figure 5.6 Mean-Square Error in Acceleration vs. Discrete Time "i."

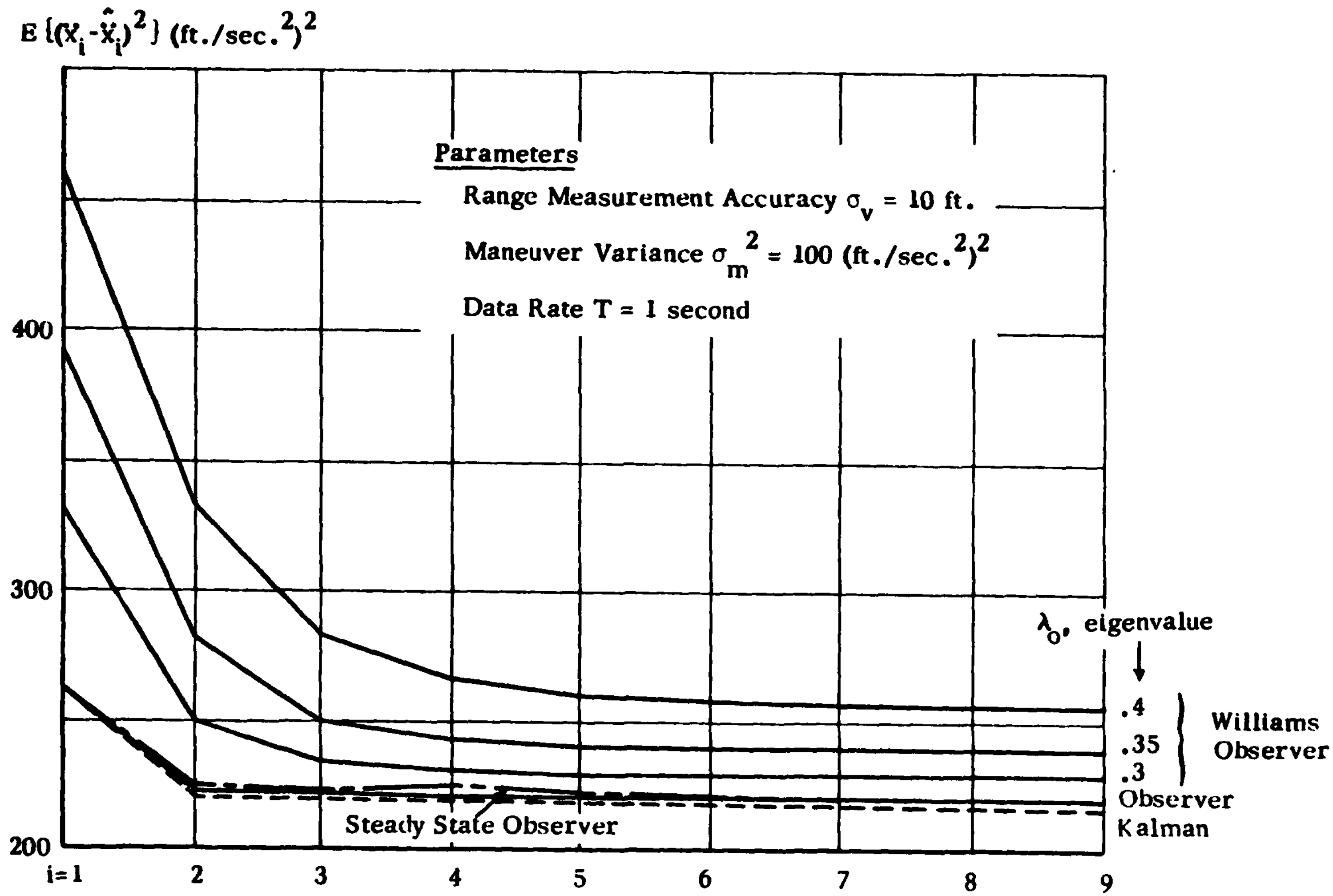


Figure 5.7 Mean-Square Error in Acceleration vs. Discrete Time "i."

5.4 EXAMPLE 2

We shall next consider the important problem of radar tracking of manned maneuvering targets as recently studied by Singer [27]. In this example the target acceleration is modeled appropriately as a time-wise correlated noise sequence of the Gauss-Markov type. The fundamental state equations describing the system in one dimension are again given by (5.1), (5.2). All the basic definitions and assumptions of example 1 are therefore assumed to hold with the exception that in example 2 the state driving noise, w_i, is taken to be a scalar Gauss-Markov sequence. Hence, w_i is obtained as the output of the discrete-time linear system

$$w_{i+1} = \rho w_i + \eta_i \tag{5.30}$$

where η_i is a zero-mean scalar white sequence with variance $\sigma_M^2(1-\rho^2)$ and ρ is the correlation between successive maneuver samples. Since w_i in (5.30) is a non-white sequence, the Kalman filter equations cannot be directly applied and it is necessary to "whiten" the input noise before the Kalman equations can be used. The usual solution to the "whitening" approach is to augment the state equations (5.1), (5.2) using the relation (5.30).
When this is done we obtain the following "augmented" state equations.

$$\underbrace{\left[\begin{array}{c} x_{i+1} \\ \dot{x}_{i+1} \\ \ddot{x}_{i+1} \\ \hline w_{i+1} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & T & \frac{T^2}{2} & 0 \\ 0 & 1 & T & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & \rho \end{array}\right] \left[\begin{array}{c} x_i \\ \dot{x}_i \\ \ddot{x}_i \\ \hline w_i \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 1 \end{array}\right] \eta_i}_{\underline{x}_{i+1}^{(a)} = A_i^{(a)} \underline{x}_i^{(a)} + \underline{w}_i^{(a)}} \tag{5.31}$$

$$y_i = [1 \quad 0 \quad 0 \mid 0] \begin{bmatrix} x_i \\ \dot{x}_i \\ \ddot{x}_i \\ \hline w_i \end{bmatrix} + v_i \tag{5.32}$$

$$y_i^{(a)} = H_i^{(a)} \underline{x}_i^{(a)} + \underline{v}_i$$

It is clear that Kalman's filter for the above augmented system is a 4-dimensional filter and to obtain the solution for Kalman's optimal weighting matrix it is necessary to solve recursively the augmented Kalman algorithms

$$\hat{\underline{x}}^{(a)}_{i+1/i+1} = A_i^{(a)} \hat{\underline{x}}^{(a)}_{i/i} + K^{(a)}_{i+1} (y^{(a)}_{i+1} - H^{(a)}_{i+1} A_i^{(a)} \hat{\underline{x}}^{(a)}_{i/i})$$

$$K^{(a)}_{i+1} = P^{(a)}_{i+1/i} H^{(a)\prime}_{i+1} (H^{(a)}_{i+1} P^{(a)}_{i+1/i} H^{(a)\prime}_{i+1} + R_{i+1})^{-1} \tag{5.33}$$

$$P^{(a)}_{i+1/i} = A_i^{(a)} P^{(a)}_{i/i} A_i^{(a)} + Q_i^{(a)}$$

$$P^{(a)}_{i+1/i+1} = (I_{n+1} - K^{(a)}_{i+1} H^{(a)}_{i+1}) P^{(a)}_{i+1/i} \tag{5.34}$$

The superscript "(a)" implies the augmented system as defined in (5.31), (5.32). In the defining equations for the augmented Kalman filter the covariance matrix for the augmented error vector is

$$P^{(a)}_{i+1/i+1} \stackrel{\Delta}{=} E\{(\hat{\underline{x}}^{(a)}_{i+1/i+1} - \underline{x}^{(a)}_{i+1})(\hat{\underline{x}}^{(a)}_{i+1/i+1} - \underline{x}^{(a)}_{i+1})'\} \tag{5.35}$$

Using the approach of Singer and Monzingo [28] we initialize the augmented Kalman filter equations by taking as the initial state estimate for the augmented state the following:

$$
\begin{aligned}
\hat{x}_{o/o} &= y_o \\
\hat{\dot{x}}_{o/o} &= \frac{1}{T}\left(\frac{3}{2}y_o - 2y_{-1} + \frac{1}{2}y_{-2}\right) \\
\hat{\ddot{x}}_{o/o} &= \frac{1}{T^2}\left(y_o - 2y_{-1} + y_{-2}\right) \\
\hat{w}_{o/o} &= 0
\end{aligned}
\tag{5.36}
$$

where again y_{-2}, y_{-1} and y_o are the first, second and third radar measurements received. The corresponding covariance initialization matrix for the augmented filter is

$$
P^{(a)}_{o/o} = \begin{bmatrix}
\sigma_v^2 & \frac{3}{2}\frac{\sigma_v^2}{T} & \frac{\sigma_v^2}{T^2} & 0 \\
\frac{3}{2}\frac{\sigma_v^2}{T} & \frac{13}{2}\frac{\sigma_v^2}{T^2} + \frac{T^2}{16}\sigma_m^2 & 6\sigma_v^2 + \frac{T}{8}\sigma_m^2 + \frac{\rho T}{4}\sigma_m^2 & \frac{\rho^2 T}{4}\sigma_m^2 \\
\frac{\sigma_v^2}{T^2} & \frac{6\sigma_v^2}{T^3} + \frac{T}{8}\sigma_m^2 + \frac{\rho T}{4}\sigma_m^2 & \frac{6\sigma_v^2}{T^4} + \frac{5}{4}\sigma_m^2 + \rho\sigma_m^2 & \rho\sigma_m^2 + \frac{\rho^2}{2}\sigma_m^2 \\
0 & \frac{\rho^2 T\sigma_m^2}{4} & \rho\sigma_m^2 + \frac{\rho^2}{2}\sigma_m^2 & \sigma_m^2
\end{bmatrix}
\tag{5.37}
$$

The two equations in (5.34) constitute the Riccati equations that must be solved at each discrete time instant to obtain the Kalman weighting matrix, $K_{i+1}^{(a)}$. For this simple example the Kalman filter computations are increased significantly due to the addition of the extra state variable introduced by way of the augmenting procedure. For example, Williams [32] has shown that the number of multiplications or additions required to solve the Kalman equations is given by the result

$$N = 3n^3 + 2mn^2 + 2m^2n + 2m^3 + n^2 + 2mn \tag{5.38}$$

where $\underline{x}_i$ is an n-vector and $\underline{y}_i$ is an m-vector. Whereas the non-augmented system (5.1), (5.2) originally required a total of N = 122 multiplications or additions (since n=3, m=1), the augmented system defined in (5.31), (5.32) requires N=258 multiplications or additions (since n=4, m=1). Clearly the Kalman filter computational requirements have been significantly increased due to the mere addition of a single state variable.

Design of the minimal-order observer for this example uses directly the "non-augmented" state equations (5.1), (5.2) together with the relation (5.30). The design procedure is described in detail in Chapter 4. Since the state equations (5.1), (5.2) are already in the desired observer canonical form, the basic observer structure for this example is identical with that obtained in the previous example 1. [See equations (5.10) through (5.14).] However, computation of the observer gain matrix, K_{i+1}, is modified appropriately to account for the non-zero cross-correlation term $\overline{\underline{\epsilon}_i \underline{w}_i'}$. That is, the optimal gain, K_{i+1}, is obtained recursively using the covariance matrix.

$$\Omega_i = A_i P_i \overline{\underline{\epsilon}_i \underline{\epsilon}_i'} P_i' A_i' + A_i V_i R_i V_i' A_i' + Q_i - A_i P_i \overline{\underline{\epsilon}_i \underline{w}_i'} - \overline{\underline{w}_i \underline{\epsilon}_i'} P_i' A_i'$$

where the cross-covariance $\overline{\underline{\epsilon}_i \underline{w}_i'}$ is computed recursively as $\overline{\underline{\epsilon}_{i+1} \underline{w}_{i+1}'} = T_{i+1}(A_i P_i \overline{\underline{\epsilon}_i \underline{w}_i'} - Q_i)\Gamma_i'$. We shall omit the unnecessary details since the solution of the gain matrix K_{i+1} is quite similar to that obtained previously in example 1. Using the notation of the previous example we obtain the optimal gain elements, $k_{i+1}^{(1)}$ and $k_{i+1}^{(2)}$, in closed form as follows.

$$k_{i+1}^{(1)} = - \frac{\omega_{12}^{\,i}}{\omega_{11}^{\,i} + \sigma_v^2} \tag{5.39}$$

$$k_{i+1}^{(2)} = - \frac{\omega_{13}^{\,i}}{\omega_{11}^{\,i} + \sigma_v^2} \tag{5.40}$$

where

$$\begin{aligned}
\omega_{11}^{\,i} &= T^2 \epsilon_{11}^{\,i} + T^3 \epsilon_{12}^{\,i} + \frac{T^4}{4} \epsilon_{22}^{\,i} \\
&\quad + \sigma_v^2 \left(1 - T k_i^{(1)} - \frac{T^2}{2} k_i^{(2)}\right)^2 \\
\omega_{12}^{\,i} &= T \epsilon_{11}^{\,i} + \frac{3T^2}{2} \epsilon_{12}^{\,i} + \frac{T^3}{2} \epsilon_{22}^{\,i} \\
&\quad - \sigma_v^2 \left(1 - T k_i^{(1)} - \frac{T^2}{2} k_i^{(2)}\right)\left(k_i^{(1)} + T k_i^{(2)}\right) \\
\omega_{13}^{\,i} &= T \epsilon_{12}^{\,i} + \frac{T^2}{2} \epsilon_{22}^{\,i} - \sigma_v^2 \left(1 - T k_i^{(1)} - \frac{T^2}{2} k_i^{(2)}\right) k_i^{(2)} \\
&\quad - T\, \epsilon w_{13}^{\,i} - \frac{T^2}{2} \epsilon w_{23}^{\,i}
\end{aligned} \tag{5.41}$$

In obtaining the above results we have used the notation

$$\overline{\underline{\epsilon}_i \underline{w}_i{}'} \triangleq \begin{bmatrix} \epsilon w_{11}{}^i & \epsilon w_{12}{}^i & \epsilon w_{13}{}^i \\ \epsilon w_{21}{}^i & \epsilon w_{22}{}^i & \epsilon w_{23}{}^i \end{bmatrix} \tag{5.42}$$

where for this example only the elements $\epsilon w_{13}{}^i$ and $\epsilon w_{23}{}^i$ are non-zero and propagate according to the relations

$$\begin{aligned} \epsilon w_{13}{}^{i+1} &= \rho\left[k_{i+1}^{(1)}\left(T\,\epsilon w_{13}{}^i + \frac{T^2}{2}\,\epsilon w_{23}{}^i\right) + \epsilon w_{13}{}^i + T\,\epsilon w_{23}{}^i\right] \\ \epsilon w_{23}{}^{i+1} &= \rho\left[k_{i+1}^{(2)}\left(T\,\epsilon w_{13}{}^i + \frac{T^2}{2}\,\epsilon w_{23}{}^i\right) + \epsilon w_{23}{}^i - \sigma_m{}^2\right] \end{aligned} \tag{5.43}$$

Initialization of the observer proceeds as follows. The initial observer error covariance matrix is of the form $\overline{\underline{\epsilon}_1\underline{\epsilon}_1{}'} = T_1 \Omega_o T_1{}'$ and it is easy to show that for this example

$$\Omega_o = A_o M_o A_o{}' + A_o \overline{(\underline{x}_o - \hat{\underline{x}}_o)\,\underline{w}_o{}'} + \overline{\underline{w}_o(\underline{x}_o - \hat{\underline{x}}_o)'}A_o{}' + Q_o \tag{5.44}$$

where $M_o = E\{(\underline{x}_o - \hat{\underline{x}}_o)(\underline{x}_o - \hat{\underline{x}}_o)'\}$ and $\hat{\underline{x}}_o$ is defined in (5.8). Evaluating M_o yields the result

$$M_o = \left[\begin{array}{c|c|c} \sigma_v{}^2 & \frac{3}{2}\,\frac{\sigma_v{}^2}{T} & \frac{\sigma_v{}^2}{T^2} \\ \hline \frac{3}{2}\,\frac{\sigma_v{}^2}{T} & \frac{13}{2}\,\frac{\sigma_v{}^2}{T^2} + \frac{T^2\sigma_m{}^2}{16} & \frac{6\sigma_v{}^2}{T^3} + \frac{T\sigma_m{}^2}{8} + \frac{\rho T\sigma_m{}^2}{4} \\ \hline \frac{\sigma_v{}^2}{T^2} & \frac{6\sigma_v{}^2}{T^3} + \frac{T\sigma_m{}^2}{8} + \frac{\rho T\sigma_m{}^2}{4} & \frac{6\sigma_v{}^2}{T^4} + \frac{5}{4}\sigma_m{}^2 + \rho\sigma_m{}^2 \end{array}\right] \tag{5.45}$$

Also, the cross-covariance $\overline{(\underline{x}_o - \hat{\underline{x}}_o)\,\underline{w}_o'}$ is found to be

$$\overline{(\underline{x}_o - \hat{\underline{x}}_o)\,\underline{w}_o'} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & \dfrac{\rho^2 T\sigma_m^2}{4} \\ \hline 0 & 0 & \dfrac{\rho\sigma_m^2}{2} + \rho\sigma_m^2 \end{array}\right] \tag{5.46}$$

Substituting (5.45) and (5.46) into (5.44) yields the initialization matrix Ω_o. Omitting the unnecessary details, the optimal initializing gain elements $k_1^{(1)}$ and $k_1^{(2)}$ are found to be

$$k_1^{(1)} = -\frac{\dfrac{21\sigma_v^2}{T} + \dfrac{7\sigma_m^2 T^3}{8}(1+\rho)}{19\,\sigma_v^2 + \dfrac{\sigma_m^2 T^4}{2}(1+\rho) + \sigma_v^2}$$

$$k_1^{(2)} = -\frac{\dfrac{10\sigma_v^2}{T^2} + \dfrac{\sigma_m^2}{4}(3 + 5\rho + 2\rho^2)}{19\,\sigma_v^2 + \dfrac{\sigma_m^2 T^4}{2}(1+\rho) + \sigma_v^2} \tag{5.47}$$

Finally, the elements ϵw_{13}^i and ϵw_{23}^i given in (5.43) are initialized as follows:

$$-\epsilon w_{13}' = \frac{T^2\sigma_m^2}{2}(\rho^2 + \rho^3)\,k_1^{(1)} + \sigma_m^2 T\left(\rho^2 + \frac{3}{4}\rho^3\right)$$

$$-\epsilon w_{23}' = \frac{T^2\sigma_m^2}{2}(\rho^2 + \rho^3)\,k_1^{(2)} + \sigma_m^2\left(\rho + \rho^2 + \frac{\rho^3}{2}\right) \tag{5.48}$$

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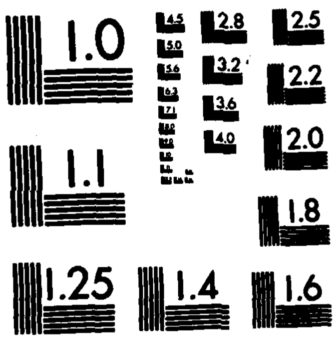

MICROCOPY RESOLUTION TEST CHART
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5.5 PERFORMANCE EVALUATION, EXAMPLE 2

In this example we have assumed the radar range measurements are independent from sample to sample and the accuracy of the range data is $\sigma_v = 10$ ft. r.m.s. For the target model we have taken the target maneuver variance to be $\sigma_m^2 = 100 \text{ (ft./sec.}^2)^2$ and the maneuver time constant (correlation) has been varied in parametric fashion. More specifically, maneuver correlations of 0, .2, .4, .6 and .8 were evaluated in the study and the tracking data rate, T, was assumed to be 1 second. In each case considered, the tracking performance of the 4-state Kalman filter and the 2-state minimal-order observer was evaluated. We shall next present the computer results shown graphically in Figures 5.8 through 5.11.

The total mean-square estimation error versus discrete time "i" for both the Kalman filter and observer is shown in Figure 5.8. Note in this figure that we have plotted trace $E\ \{(x_i - \hat{x}_i)(x_i - \hat{x}_i)'\}$ versus "i" and therefore the Kalman filter curves do not contain the error contribution in estimating the augmented state variable, w_i. Referring to Figure 5.8 it is seen that the total steady-state mean-square estimation error for the observer is increased from that of the Kalman filter by 5.9%, 5.17%, 5.0%, 6.6% and 16.5% for target maneuver correlations of 0, .2, .4, .6 and .8 respectively. These results indicate that the overall tracking performance is dependent upon ρ, the maneuver correlation, as is expected. From the viewpoint of tracking system design, however, it is more meaningful to consider the individual accuracies in target position and velocity estimates since these two quantities are the critical design quantities. For this reason we have shown in Figures 5.9 and 5.10, respectively, the mean square errors in target position and velocity. From Figure 5.9 it is seen that Kalman filtering

improves the initial measurement accuracy of 10 ft. r.m.s. to, at best, about 9 ft. r.m.s. in the steady state. As stated previously in example 1, this slight improvement in position accuracy is hardly worth the increase in numerical and computational complexities associated with mechanizing the 4-state Kalman filter. The corresponding mean-square errors in target velocity are shown in Figure 5.10. From these curves it is determined that the steady-state accuracy loss in the velocity estimate (ft./sec.) incurred in using the 2-state observer instead of the 4-state Kalman filter is approximately 3.3%, 3.1%, 3.2%, 3.6% and 6.9% for maneuver correlations of 0, .2, .4, .6 and .8, respectively. For completeness, we have also included, in Figure 5.11, the corresponding mean-square error in the estimate of target acceleration.

Table 1 shows the parametric behavior of the optimal observer gain elements, $k_i^{(1)}$ and $k_i^{(2)}$, versus discrete time "i" for each of the maneuver correlations considered. The purpose of including Table I in this example is to point out the time-varying nature of the optimal observer solution which, of course, is also a fundamental property of the Kalman filter. After an initial transient period, the error covariance matrices settle down and remain constant and likewise the corresponding optimal observer gain elements remain constant. This same phenomenon occurs in Kalman filtering theory for problems where the system matrices (A, H) are time-invariant and the noise inputs are stationary stochastic sequences. In examining Table 1 it is interesting to note that, generally speaking, the magnitude of the observer gain increases as the correlation increases from $\rho = 0$ to $\rho = .8$. Also, from Table 1 it is seen that the observer settling time tends to increase as the maneuver time constant increases. The settling time of the observer

is comparable, however, with that of the Kalman filter, as can be seen in Figures 5.8 through 5.11.

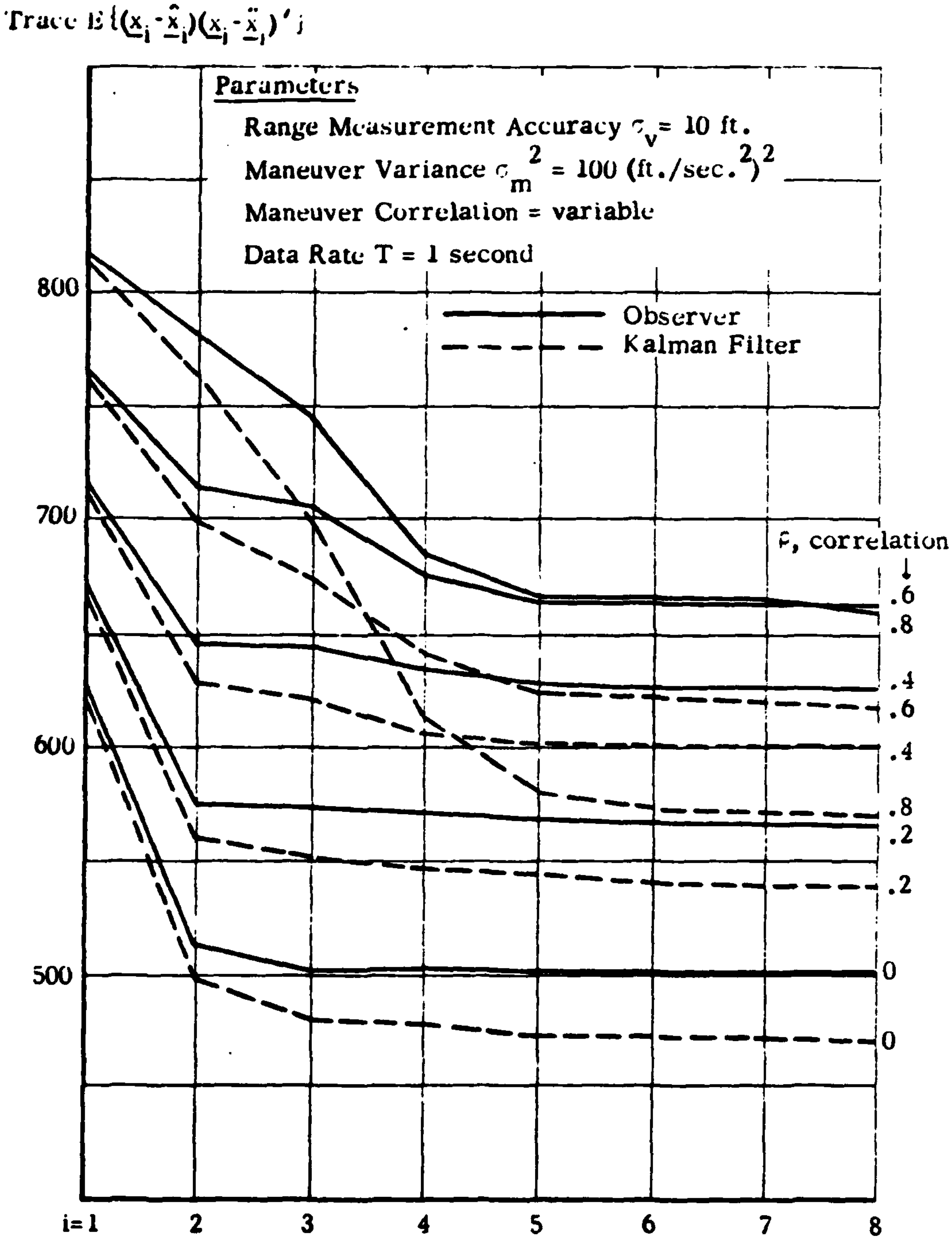

Figure 5.8 Total Mean-Square Estimation Error vs. Time "i."

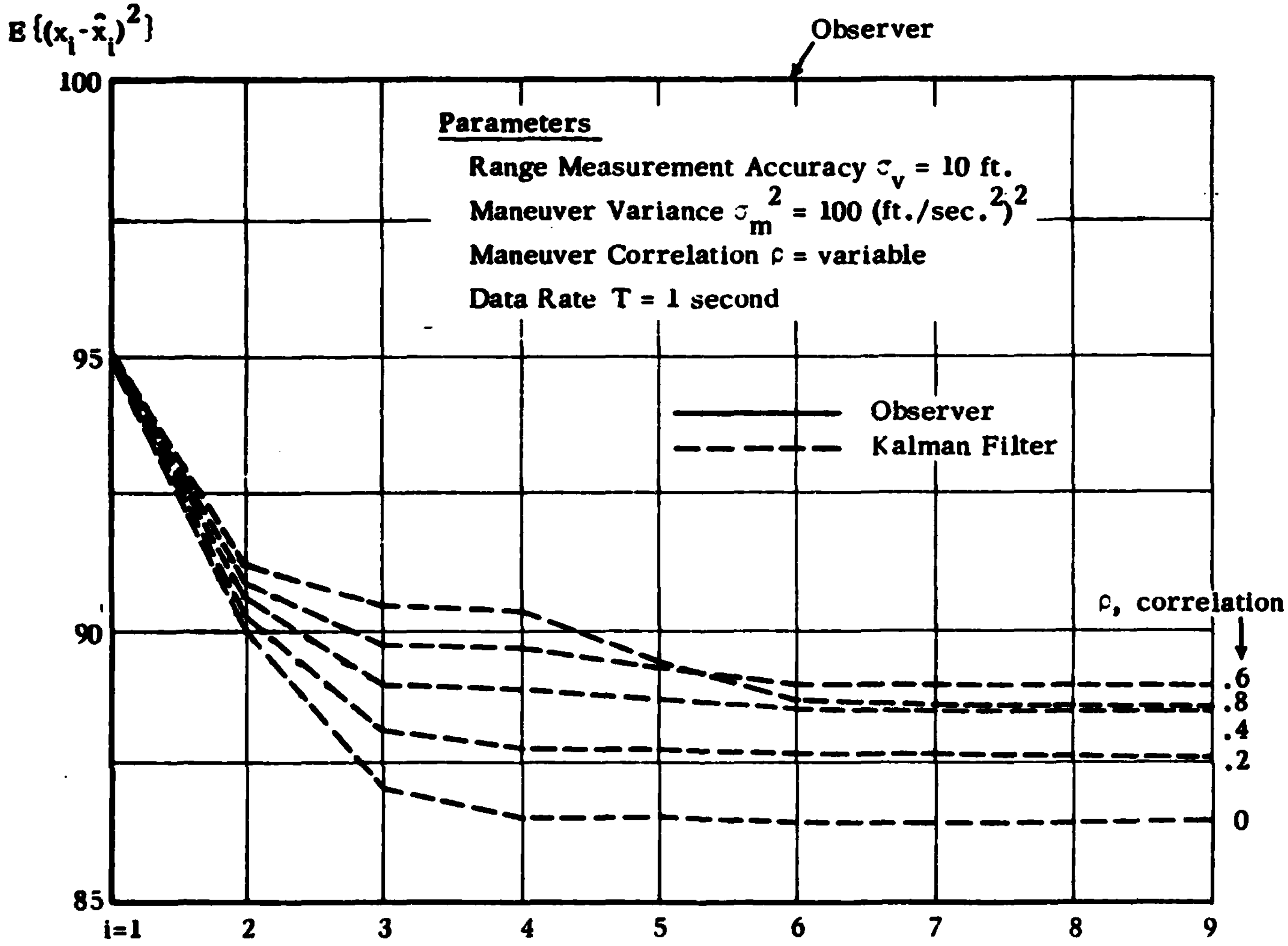

Figure 5.9 Mean-Square Error in Position vs. Time "i."

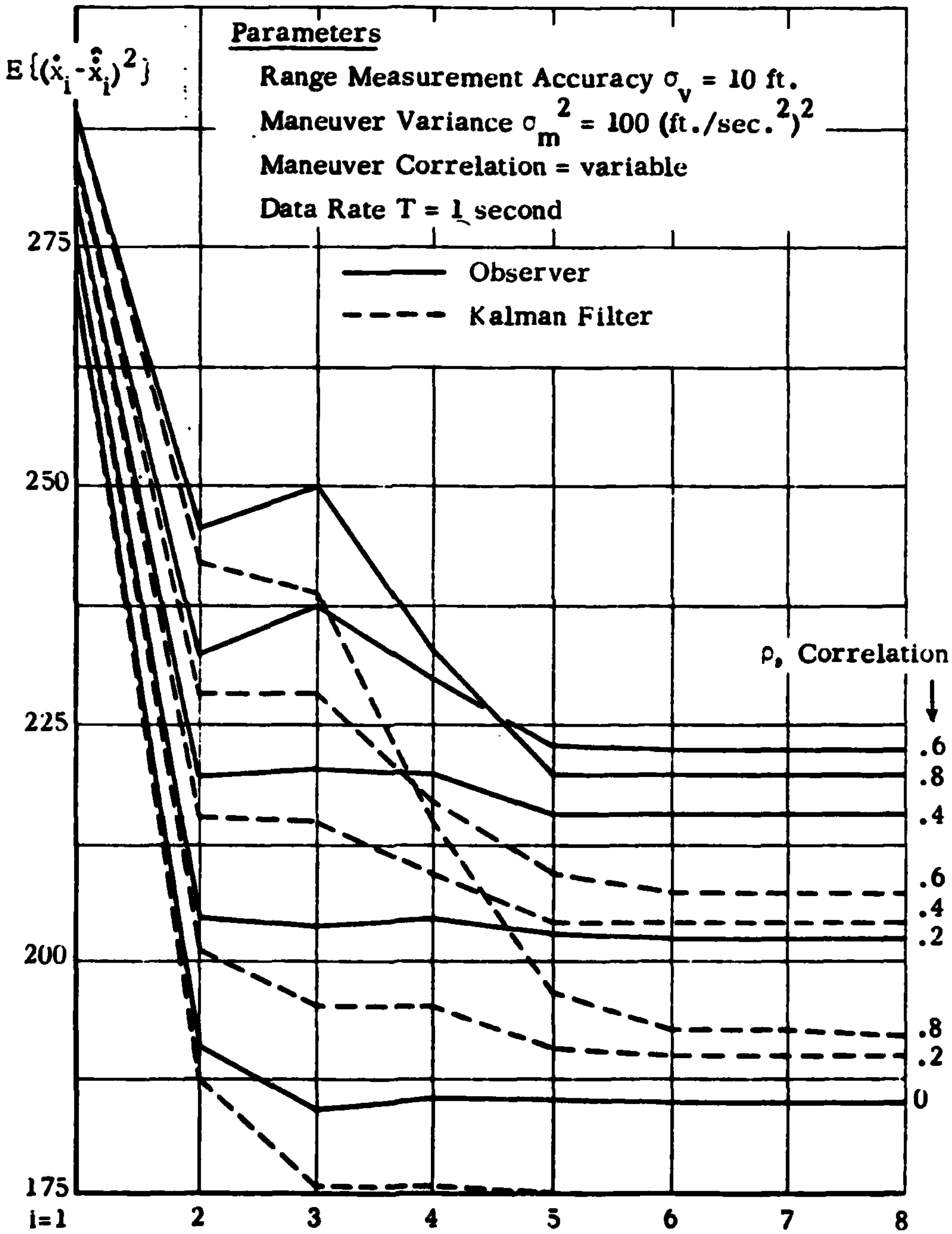

Figure 5.10 Mean-Square Error in Velocity vs. Time "i."

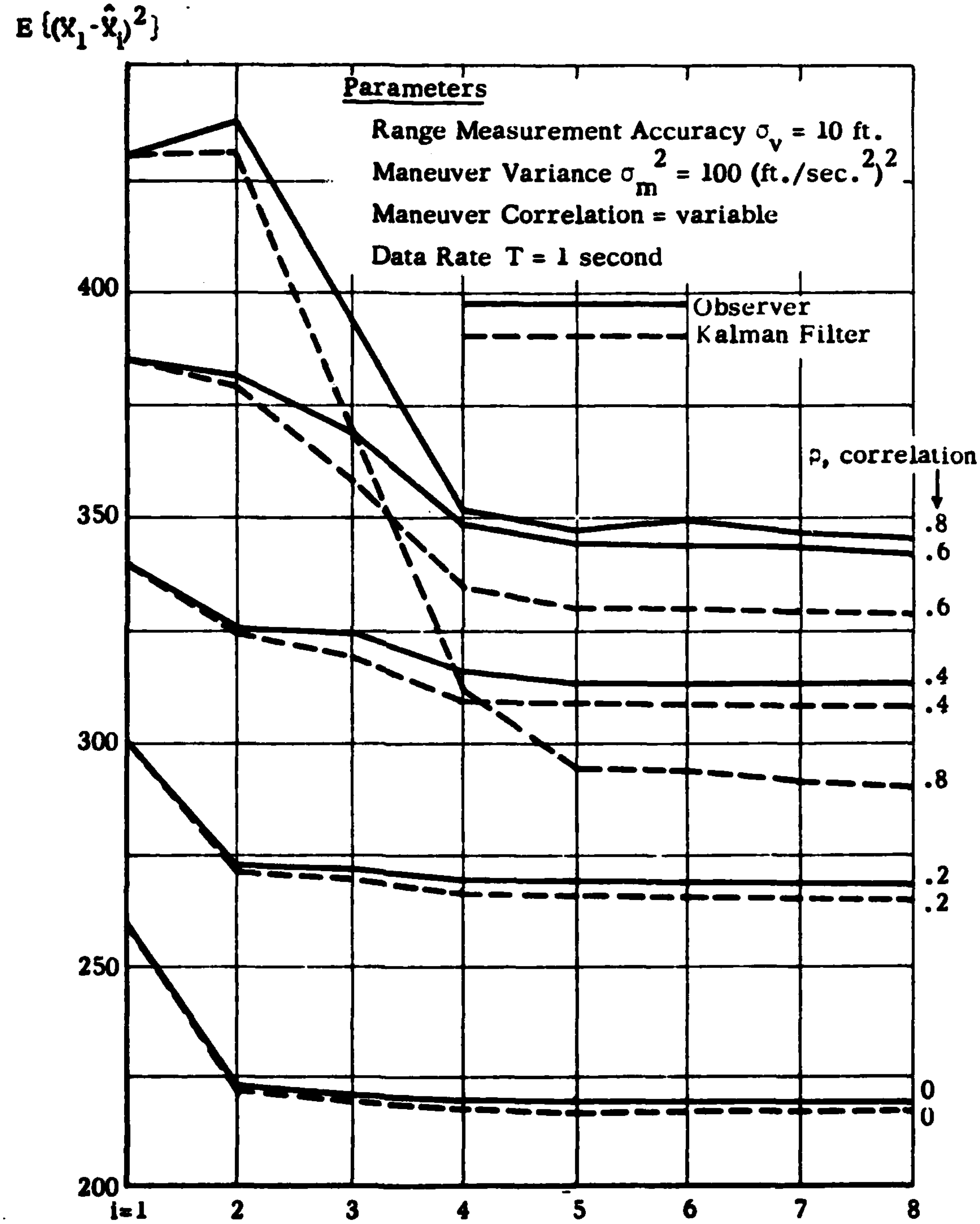

Figure 5.11 Mean-Square Error in Acceleration vs. Time "i."

Table 5.1. Observer Gains $k_i^{(1)}$ and $k_i^{(2)}$ Versus "i."

	$\rho=0$		$\rho=.2$		$\rho=.4$		$\rho=.6$		$\rho=.8$	
	$k_i^{(1)}$	$k_i^{(2)}$	$k_i^{(1)}$	$k_i^{(2)}$	$k_i^{(1)}$	$k_i^{(2)}$	$k_i^{(1)}$	$k_i^{(2)}$	$k_i^{(1)}$	$k_i^{(2)}$
1	-1.0671	-0.5244	-1.0704	-0.5350	-1.0737	-0.5473	-1.0769	-0.5615	-1.0801	-0.5775
2	-0.8529	-0.3759	-0.8739	-0.4127	-0.8954	-0.4571	-0.9172	-0.5093	-0.9389	-0.5691
3	-0.7925	-0.3569	-0.8365	-0.4146	-0.8734	-0.4804	-0.9171	-0.5529	-0.9517	-0.6299
4	-0.7926	-0.3627	-0.8416	-0.4214	-0.8835	-0.4845	-0.9169	-0.5488	-0.9398	-0.6084
5	-0.7941	-0.3622	-0.8398	-0.4185	-0.8769	-0.4784	-0.9027	-0.5377	-0.9130	-0.5876
6	-0.7926	-0.3610	-0.8377	-0.4176	-0.8741	-0.4783	-0.8987	-0.5386	-0.9068	-0.5915
7	-0.7923	-0.3611	-0.8378	-0.4181	-0.8745	-0.4790	-0.8993	-0.5399	-0.9092	-0.5952
8	-0.7926	-0.3613	-0.8380	-0.4182	-0.8744	-0.4788	-0.8988	-0.5392	-0.9082	-0.5934
9	-0.7926	-0.3613	-0.8379	-0.4181	-0.8742	-0.4787	-0.8983	-0.5390	-0.9067	-0.5925
10	-0.7926	-0.3612	-0.8379	-0.4181	-0.8742	-0.4788	-0.8983	-0.5392	-0.9067	-0.5930
11	-0.7926	-0.3613	-0.8379	-0.4181	-0.8742	-0.4788	-0.8983	-0.5392	-0.9068	-0.5931
12	-0.7926	-0.3613	-0.8379	-0.4181	-0.8742	-0.4788	-0.8983	-0.5391	-0.9067	-0.5929
13	-0.7926	-0.3613	-0.8379	-0.4181	-0.8742	-0.4788	-0.8983	-0.5391	-0.9066	-0.5929
14	-0.7926	-0.3613	-0.8379	-0.4181	-0.8742	-0.4788	-0.8983	-0.5391	-0.9066	-0.5929
15	-0.7926	-0.3613	-0.8379	-0.4181	-0.8742	-0.4788	-0.8983	-0.5391	-0.9066	-0.5929
16	-0.7926	-0.3613	-0.8379	-0.4181	-0.8742	-0.4788	-0.8983	-0.5391	-0.9066	-0.5929
17	-0.7926	-0.3613	-0.8379	-0.4181	-0.8742	-0.4788	-0.8983	-0.5391	-0.9066	-0.5929
18	-0.7926	-0.3613	-0.8379	-0.4181	-0.8742	-0.4788	-0.8983	-0.5391	-0.9066	-0.5929
19	-0.7926	-0.3613	-0.8379	-0.4181	-0.8742	-0.4788	-0.8983	-0.5391	-0.9066	-0.5929

Parameters:

Radar Measurement Accuracy, $\sigma_v = 10$ (ft.)
Target Maneuver Variance, $\sigma_m^2 = 100$ (ft/sec^2)^2
Target Maneuver Correlation, ρ = variable
Data Rate, T = 1 second

6. SUMMARY AND SUGGESTIONS FOR FURTHER WORK

6.1 SUMMARY AND CONCLUSIONS

This dissertation has considered the problem of estimating the state of a linear time-varying discrete system using an observer of minimum dynamic order. In Chapter 3 of the dissertation we consider systems for which the plant noise $\underline{w}_i$ and measurement noise $\underline{v}_i$ are modeled as Gaussian white sequences. The effects of these noise disturbances upon the estimation error are considered as an integral part of the fundamental development. The solution of the observer design uses a special linear transformation which transforms the given state equations into an equivalent state space which is extremely convenient from the standpoint of observer design. Design of the observer is then based on a special observer configuration containing a free gain matrix, K_i, which is chosen to minimize the mean-square estimation error at time "i." The solution obtained is optimal at each instant "i" and therefore is optimal both during the transient period and in the steady state. Computation of this gain matrix is done recursively as in the Kalman filter algorithms, however, computationally the solution is much simpler than for the Kalman filter. In the special case of no measurement noise, the observer estimation errors are identical with that of the corresponding Kalman filter. The main contribution of Chapter 3 is, therefore, the development of a completely unified theory for the design of optimal minimal-order observers applicable to both time-varying and time-invariant discrete systems for which the plant noise $\underline{w}_i$ and measurement noise $\underline{v}_i$ are modeled as Gaussian white sequences.

In Chapter 4 we have extended the basic optimal minimal-order observer theory to cover that class of systems for which the noise disturbances $\underline{w}_i, \underline{v}_i$ are time-wise correlated and are modeled adequately as Gauss-Markov processes. The usual approach to this estimation problem is to augment the state vector and design the estimator (be it a Kalman filter, observer, etc.) to provide estimates of the total augmented state. In Chapter 4 we have utilized the basic observer structure developed in Chapter 3 and have modified the observer gain matrix appropriately to obtain minimum mean-square estimates of the plant states without an increase in the dimension of the observer (i.e., the observer dimension remains "n-m"). Along similar lines, we have also considered the special case whereby the plant noises $\underline{w}_i$ and $\underline{v}_i$ are white sequences which are crosscorrelated at time "i" (that is, $E\{\underline{w}_i \underline{v}_j{}'\} = S_i \delta_{ij}$) and have modified the observer gain matrix appropriately to provide optimal performance in the mean-square sense.

To illustrate the typical application of the observer designs developed in this dissertation we have considered, in Chapter 5, the design of a radar tracking system. In the first example we treat the situation where the noises $\underline{w}_i$ and $\underline{v}_i$ are white sequences. In this example, the performance of the optimal minimal-order observer is compared with that of other estimators including the Kalman filter and also several equal-eigenvalue observer designs. It is shown that for a typical set of radar and target model parameters the optimal observer provides extemely good tracking performance and is superior by far to the equal eigenvalue designs of Dellon [10] and Williams [32]. I example ? we treat the situation where the target acceleration is mo... ed as a time-wise correlated noise sequence. Here the

2-dimensional optimal observer is compared with the corresponding 4-state Kalman filter and it is shown that the observer provides acceptable tracking performance over a wide spectrum of target maneuver time constants. The examples of Chapter 5 clearly illustrate the practicality of the observer design techniques developed in the dissertation.

6.2 TOPICS FOR FUTURE INVESTIGATION

During the course of performing this research several closely associated unsolved problems of an extremely fundamental nature have been uncovered and these problems might form the basis for further research. In this dissertation we have considered only observers of minimal dynamic order. That is, the dimension of the dynamical portion of the estimator is "n-m" where "n" is the dimension of the state vector to be estimated and "m" is the number of independent available outputs. Since it has been demonstrated quite vividly that the Kalman filter is an observer of dimension "n" and since the Kalman filter provides the best performance in terms of minimizing the mean-square estimation error, the idea of considering non-minimal order observers is appealing. (A non-minimal order observer has dynamic order greater than the minimal order observer but less than the Kalman filter.)

It is conjectured that through the use of non-minimal order observers the estimation error can be reduced even further from that attained with the optimal minimal-order observer developed in this dissertation. However, the improvement in estimation accuracy is undoubtedly accomplished only at the cost of increased complexity. This non-minimal order observer would have important application in the class of systems where some of the outputs are relatively noise-free while the remaining outputs are rather noisy and

must be filtered. In this proposed domain of research the literature is completely lacking and therefore it is recommended that further work be done along these lines.

Another area of research which appears to be relatively void of investigation is in the area of super low-order observers. When an estimate of some fixed linear combination of states is required, it is well known [21] that such an estimate can be obtained using an observer of order less than the minimal order, "n-m." A consideration of the effects of system noise inputs upon the performance of these so called super low-order observers may lead to an optimal design similar to the optimal minimal-order observer developed in this dissertation.

Another possible topic for future research of a more practical nature is the design of observers via the selection of observer eigenvalues. To date, most of the literature pertaining to the design and optimization of observer systems has been concerned with the ability to specify, with complete freedom, the choice of observer eigenvalues. In fact, numerous researchers have been able to demonstrate through the clever use of special canonical forms that it is possible to design observers with completely arbitrary eigenvalues provided the plant equations satisfy the observability criterion. However, it is clear that without a thorough analysis of the effects of noise the question of where to optimally place the observer eigenvalues for reasonable performance is still unanswered and remains a perplexing problem to the systems designer. It is one thing to be able to design observer systems with complete freedom in the choice of observer eigenvalues, but it is another to be able to specify what the eigenvalues should be. Very little has been written about this latter aspect.

In the design of an observer for any given fixed plant, one possible approach to this eigenvalue selection problem might be to first investigate the eigenvalues of the corresponding Kalman filter in order to establish some guidelines for selecting the observer eigenvalues. In restricting the class of admissible observers to be investigated, some fundamental rules might be developed for the optimal choice of observer eigenvalues. Results of a fundamental nature are also lacking in the domain of adaptive observer design wherein the choice of observer eigenvalues is modified with time in an optimal fashion according to the noise statistics, signal to noise ratio, or some other criterion. Much research remains to be done in the domain of observer eigenvalue selection where the minimization of noise effects upon system performance is of prime importance.

Finally, it should be mentioned that the design of the optimal minimal-order observer for time-varying continuous-time systems is still an unsolved problem. This problem was investigated by Ash [4, 5] who considered the design of a minimal-order observer for continuous time-varying linear systems (i.e., the continuous-time analog of the discrete-time problem treated in this dissertation) with the goal of obtaining an observer design which minimized the effects of noise upon the estimation accuracy of the observer derived estimates. Ash proposed a suboptimal trial-and-error-type solution to the problem and hence the results of his work are an "engineering" rather than a "mathematical" solution to the design problem. He was unable to select the free gain matrix $K(t)$ (analogous to the free gain matrix K_i of the discrete observer) to absolutely minimize the overall-mean-square estimation error. It is conjectured that an optimal-observer gain matrix, $K^*(t)$, does exist for the continuous time-varying minimal-order observer

and the solution of this problem would be an important contribution to the theory.

One possible approach to the solution of this problem might be to discretize the continuous time-varying state equations obtaining a model for the plant which is valid at discrete intervals Δt seconds apart. Then the theory developed in this dissertation for discrete-time systems may be applied to the discrete representation of the plant and the optimal discrete minimal-order observer derived. Taking the sample interval Δt sufficiently small one would obtain a reasonably good approximation to the continuous time problem. Of course, it is of interest to obtain a closed form solution for the optimal observer gain $K^*(t)$ analogous to the gain K_i^* of the discrete time observer. In [15] Kalman was able to obtain the continuous-time Kalman filter solution from a consideration of the discretized model by taking the limit as $\Delta t \rightarrow 0$. But even Kalman himself questioned the rigor of this approach and in [18] Kalman took a more rigorous approach and solved the Wiener-Hopf equation directly to obtain the continuous time version of the Kalman filter. It is recommended that further work be done in the area of minimal-order observer design for continuous-time equations with the goal of determining the optimal time-varying solution.

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